7. In his second paper, Professor Allen obtained the coördinates of the foci, the equations of the directrices, and the eccentricity of a conic section directly from the general equation of the second degree without change of coördinate axes.

8. In his third paper, Professor Allen gave simple geometric methods for finding 16 tangents to an ellipse and their points of tangency, when only the semi-axes are known.

9. Professor McGaw discussed the problem of teaching mathematics with a view to reviving interest in its study. He suggested several means of attaining this interest, such as (*a*) the use of models, slide rules, paradoxes, etc., (*b*) the encouraging of mathematics clubs, and (*c*) the organization of a system of popular lectures designed to convince people that mathematics has a very vital and continuous contact with everyday affairs.

10. In his first paper Professor Reilly emphasized the fact that the isobaric differential equation of the first order and the first degree can be solved as easily as the corresponding homogeneous equation, and recommended that it be included in a first course in differential equations.

11. In his second paper Professor Reilly pointed out that for interpolating values of a function between two given values y_1 and y_2 there exists an infinite number of formulas dependent upon the type of curve assumed and upon additional conditions that may be imposed. Discretion in imposing conditions and simplicity of formula obtained were given as two criteria of a good interpolation formula. One such formula was developed.

12. In his first paper Professor Turner proved that if P and Q are any two points on a great circle passing through any point O on the sphere, and if PM and QN are great circle arcs drawn perpendicular to any great circle through O, then

$$\frac{\sin PQ}{\sin MN} = \frac{\sin OP}{\sin OM} \cos NQ.$$

As an application it was shown that if T be any point on the tangent at P to a spherical conic of which S is a focus, and if TM be drawn perpendicular to SP, and TN perpendicular to the directrix, then

$$\frac{\sin SM}{\sin TN} = \frac{e}{\cos TM},$$

where e is the eccentricity of the conic.

13. In his second paper Professor Turner showed that if AB is any chord of a circle, center O, and if P and Q are any pair of inverse points, then $AP \cdot BQ = AO \cdot QR$, where R is the symmetric point (reflection) of P with respect to AB. Some applications of the theorem were stated.

14. Professor Woods derived the form of the equation of the S_{n-1}, in a space of n dimensions S_n, on an S_{n-2} determined by two S_{n-1}'s, and on an S_0 determined by n S_{n-1}'s. He then found the harmonic conjugate of this S_{n-1}, and discussed the cases $n = 2$ and 3. As an application he showed how this theorem enables

one to write down easily the equation of the quadric as the locus of a line that always touches three given skew lines in space.

15. Professor Smith in his paper showed that a q-difference equation of the form

$$p_1(x)\varphi(x) + p_2(x)\varphi(qx) + p_3(x)\varphi(q^2x) = 0,$$

where $\varphi(x)$ is an unknown function, and $p_1(x)$, $p_2(x)$ and $p_3(x)$ are rational functions, may, with certain restrictions, be transformed by the substitution

$$f(x) = \varphi(x) \prod_{r=0}^{\infty} p_1(q^r x)$$

into the equation

$$f(x) = p_2(x)f(qx) + p_3(x)p_1(qx)f(q^2x) = 0,$$

which is satisfied by an infinite determinant $D(x)$. The convergence of $D(x)$ may be shown by elementary means and the method and results extended to similar equations of higher order.

16. If a linear point transformation is applied to the differential equation of Pearson's frequency distributions

$$\frac{dy}{dx} = \frac{(x+a)y}{b_0 + b_1x + b_2x^2},$$

under the usual condition of preservation of areas, Professor Snedecor showed that the form of the equation is invariant, together with the constants of the distribution. The relation of this transformation to the standardization of grades was discussed.

J. F. REILLY, *Secretary-Treasurer.*

THE MATHEMATICAL THEORY OF ECONOMICS.[1]

By G. C. EVANS, Houston, Texas.

1. Introduction. One interest in research in the mathematical theory of economics is that the necessary preparation for it either in mathematics or in economics is not so great as for theoretical research say in physics or chemistry, or even in biology. And yet even if one regards mathematics as the study of quantity merely, rather than of order in general, the fact that every financial transaction is controlled by numbers would indicate that if there is any theoretical science in economics, it must be a mathematical one. In fact it may well be that it is the lack of mathematical technique among economists which has prevented the theoretical side of the subject from developing as rapidly as the wealth of books and papers, devoted to it, would seem to indicate. On the other hand, if we turn to the trained mathematicians, we find them mainly

[1] Read at the annual meeting of the Association at Washington, D. C., Dec. 31, 1924.

engrossed in the more romantic fields of physics, chemistry and engineering, except in the case of the extensive analysis of statistics, where contact is made with kinetic theory on the one hand, and social and biological data on the other.

2. Monopoly and Competition. Augustine Cournot laid the foundations of this subject in 1837, and his contributions, although elementary in method, have dwarfed those of most other workers. The notable exception is Irving Fisher. In fact, the first requirements for study in our subject are to read Cournot's *Mathematical Researches into the Theory of Wealth*, and Irving Fisher's *The Purchasing Power of Money*. With the latter's investigation of the equation of exchange and its important practical consequences we have not time to deal here, but must confine ourselves to one or two special problems suggested by the former. Cournot discussed the problem of economic equilibrium, describing the steady states which, under various conditions of monopoly and competition, would make profit a maximum; inventing a theory of foreign exchange; and outlining a measure of community wealth.

If the cost of u units of a commodity is $q(u)$ and the demand is $u = \varphi(p)$, where p is the price, then in a state of equilibrium the cost is $q(\varphi(p))$, the profit is $\pi = up - q = p\varphi(p) - q(\varphi(p))$, and in order to make the profit a maximum, the equation

$$\frac{d}{dp}\{p\varphi(p) - q(\varphi(p))\} = 0 \tag{1}$$

must hold. Such is in brief Cournot's theory of monopoly. And the same sort of analysis, if there are several producers and several cost functions, will develop a theory of competition. Cournot points out the relations between the prices of equilibrium for these various kinds of production.

In an article in the MONTHLY (*1922*, 371–380), I showed that these theories could be made even simpler and more instructive by taking a simple approximate form of the cost function,

$$q = Au^2 + Bu + C, \tag{2}$$

where A, B, C are > 0, where u is the amount produced per unit time, and q the cost per unit time of producing that amount; and a simple approximate form,

$$y = ap + b, \qquad a < 0,\ b > 0, \tag{3}$$

for the amount that would be consumed in the market per unit time if the price were p. In this way the equation equivalent to (1) yields the value

$$p_m = \frac{b - 2Aab - Ba}{-a(2 - 2Aa)}.$$

If there are n producers, all with the same cost function $q(u_i) = Au_i^2 + Bu_i + C$, and the steady state is indicated by writing $y = u_1 + u_2 + \cdots + u_n$ in (3), the profit for each producer will be

$$\pi_i = pu_i - q(u_i).$$

Then under the hypothesis that each producer tries to make his profit a maximum by varying his production, independently of the others (which is Cournot's definition of competition), we get from the equation analogous to (1),

$$p_a = \frac{b - 2Aab - nBa}{-a(n + 1 - 2Aa)}.$$

On the other hand if each regulates his production so as to make the total profit a maximum, we have what may be called coöperation, with an equilibrium price

$$p_c = \frac{nb - 2Aab - nBa}{-a(2n - 2Aa)},$$

while if each tries to regulate his production so as to make the biggest profit he can, assuming that the price will not depend on his own manipulation, there results the price

$$p_b = \frac{2Ab + nB}{n - 2Aa}.$$

It is easily verified that $p_m > p_c > p_a > p_b$. Rather than obtain these results again now, let me refer for them to the article in the MONTHLY, and turn to new details.

The statement is sometimes advanced that altruism is not a social benefit, since if each looks after the other it is the same as each looking after himself by a roundabout and inefficient process. However that may be, it is assumed to be of benefit to society to have the production of a typical commodity large, and of benefit to the producer to make his profit large. As a criterion of a somewhat altruistic society then, we might seek to make not π a maximum but, say, $\pi + k^2u$, where k^2 is some convenient constant. Equation (1) becomes

$$\frac{d}{dp}\{up - Au^2 - (B - k^2)u - C\} = 0,$$

so that (by changing B to $B - k^2$) we have

$$p_m = \frac{b - 2Aab - (B - k^2)a}{-a(2 - 2Aa)}, \tag{4}$$

a lower price than before, since $a < 0$. A calculation in terms of elementary algebra shows that the amount produced would be, by the quantity

$$\delta u = \frac{-ak^2}{2 - 2Aa} > 0,$$

more than before, the profit less than before by an amount

$$-\delta\pi = \frac{|k^4a|}{4(1 - Aa)}$$

and the profit plus k^2u greater than the original profit by

$$\frac{k^2}{4(1 - Aa)}\{2(b + Ba) - k^2a\}.$$

By means of this kind of analysis a satisfactory rate might be established for a public utility, public utilities being in fact monopolies in which cost functions may be accurately known, and demand functions established by a single change of rates. If we compare (4) with the formula (23) of the MONTHLY article already mentioned, which gives the price such that, when it is fixed, the monopolist will make his greatest profit by meeting the demand:

$$\frac{b - 2Aab - (B - k^2)a}{-a(2 - 2Aa)} = \frac{B + Ab}{1 - Aa},$$

we get a kind of maximum value for k^2:

$$k^2 = \frac{b + Ba}{-a}.$$

If p is given a value not less than that given by (4) with this value of k^2, it will be to his advantage to meet the demand.

A further investigation possible in terms of elementary algebra, besides its slight foundation in the calculus, would be the discussion of coöperation and competition in the case where the cost functions of the various producers are not all identical but have coefficients A_i, B_i, C_i which differ from A, B, C by infinitesimals δA_i, δB_i, δC_i. What are then the changes in price and production if the cost coefficients of a single producer are changed by amounts δA, δB, δC? In this way may be studied, as Cournot has pointed out, the influence on the commodity of special taxes and local conditions.

The general problem of taxation, even with respect to a particular commodity, is more difficult; for the general price level enters into the problem as well as the price of the particular commodity. The problem thus introduces the price index as an additional variable, with regard to which some hypotheses must be made as to the coefficients a, b of the demand function. But the investigation is not the less worth while. Although nothing is more certain than death and taxes, nothing is less certain than their incidence.

3. Monopoly and Competition. General Problems. An extensive mathematical development of the theory arises if we no longer assume that the situation is the "steady state." We are then led into problems such as the one which the author has tried to state in the simplest possible terms in an article in the MONTHLY (*1924*, 77–83) on the Dynamics of Monopoly: to make a maximum the total profit over the interval of time during which the prices are changing, assuming that the rate of production $u(t)$ is kept equal to the amount which would be absorbed by the market. In mathematical terms:

$$\pi = \int_{t_0}^{t_1} (up - q)dt \qquad (5)$$

is to be a maximum. More general theories may well be considered, in which reserves of stock enter in, as in an article which the author has submitted to the *Proceedings of the National Academy*. But one great interest in this type of problem depends on what is taken as the law of demand. If for instance we write $u = y = ap + b$, the problem reduces to that of equilibrium already considered. If however we write

$$y = ap + b + h\frac{dp}{dt} = u, \qquad h > 0,\ b > 0,\ a < 0, \tag{6}$$

so that demand is greater if the rate of change of price is positive, other things being equal, the problem is most practical but quite different from the classical theories.

An advance, which would not involve further mathematical difficulties but would be again of considerable practical interest, would be to make use of a demand law of the form

$$y = u = ap + (b + b' \cos kt) + h\frac{dp}{dt}, \qquad b' < b,$$

which would enable us to consider seasonal changes. The reader will gain considerable familiarity with our problem if he will work out this case.

The most interesting type of law of demand has been suggested recently by Irving Fisher in a statistical investigation by means of which he establishes a correlation between the total volume of trade $T(t)$ at time t and the rate of change of price index at a previous time $t - C$, the interval of time being constant and about equal to two weeks. Professor Bray finds that the correlation is even considerably better for a formula of the type

$$T(t) = ap(t - c) + b + hp'(t - c).$$

Does not this suggest an interesting study of the integral (5)?

Another study would involve a still more general type of law of demand, say of the form

$$u(t) = ap(t) + \int_{-\infty}^{t} \varphi(t - T)p(T)dT. \tag{7}$$

This study has been initiated by C. F. Roos in an article to be published in the *American Journal of Mathematics*. As might be expected, a formula of this kind introduces integral equations into the analysis. Mr. Roos also considers a problem of competition which involves a treatment of several integrals like (5). It may be remarked that relations which arise from integral equations like (7) can often by differentiation be analyzed in terms of differential equations with simple coefficients. But we have devoted already space enough to this general problem.

4. Foreign Exchange. Cournot considered in some detail the problem of foreign exchange. For simplicity we suppose that each market makes prices with respect to a unit quantity of gold the same in all markets—say a dollar of 25 grains—because, of course, prices in other denominations can be evaluated in terms of a change of units; and we shall limit ourselves to three markets.

Let M_{ij} be the amount of money owed by market i to market j, and c_{ij} be the value for exchange purposes of a dollar of money in i in terms of credit at j. That is, suppose one dollar at i will buy c_{ij} dollars worth of credit at j. Cournot's hypothesis for the determination of rates of exchange c_{ij} is that credits shall be balanced without the transportation of money. That is, for the first market

$$M_{12} + M_{13} = c_{21}M_{21} + c_{31}M_{31}, \tag{8.1}$$

and similarly

$$M_{21} + M_{23} = c_{12}M_{12} + c_{32}M_{32}, \tag{8.2}$$

$$M_{31} + M_{32} = c_{13}M_{13} + c_{23}M_{23}. \tag{8.3}$$

Incidentally, the possibility of speculation maintains very approximately the equations

$$\begin{aligned} c_{ij}c_{ji} &= 1, \\ c_{ij}c_{jk} &= c_{ik}. \end{aligned} \tag{9}$$

By means of (9) all the rates of exchange may be expressed in terms of c_{21} and c_{31} so that the equations (8) become the following:

$$-c_{21}M_{21} + c_{31}M_{31} = M_{12} + M_{13}, \tag{10.1}$$

$$c_{21}(M_{21} + M_{23}) - c_{31}M_{32} = M_{12}, \tag{10.2}$$

$$-c_{21}M_{23} + c_{31}(M_{31} + M_{32}) = M_{13}. \tag{10.3}$$

Here (10.1) − (10.2) = (10.3), so equation (10.3) may be disregarded; and there are left two equations to determine the two unknowns.

The mathematician, rather than the economist, will now stop to consider the exceptional cases where one or more two-order determinants vanish. Since the M_{ij} are essentially ≥ 0, these resolve themselves into cases where (*a*) one or more of the markets find themselves without credit with respect to others, or (*b*) the markets split into groups which have no dealings with each other, the latter case being that in which the augmented matrix of (10) is of rank < 2. Professor Bray in an article in the MONTHLY (*1923*, 365–371) has considered the corresponding situation for an arbitrary number of markets.

When the elementary problems have been considered, the practical nature of more extended mathematical investigation becomes manifest. Thus it is opportune to replace the equations (8) by differential relations, which emphasize the rate of increase of credit, or by integral relations, which emphasize the past history of credit. The persistence of equations (9) adds an unusual element to the mathematical problem, as one can see even in the simplest case of two markets.

5. General Points of View. There is no such measurable quantity as "value" or "utility" (with all due respect to Jevons, Walras and others) and there is no evaluation of "the greatest happiness for the greatest number"; or, more flatly,—there is no such thing. In a way, material happiness has to do with a maximum of production and a minimum of unpleasant labor; though again no such thing is realizable theoretically without an arbitrary definition of a composite function which is to take on a maximal value; and in the composition of this function the labor and profit of various classes of people enter capriciously. One might define a ratio of weighted production divided by weighted amounts of labor, according to classes, and study what sort of lash, economic or otherwise, would serve to impel Society towards this limit; but the choice of weights would depend essentially on whether the chooser is born a Bolshevik or a member of the Grand Old Party! Compromises carry us into the field of ethics.

That does not mean that such study is unprofitable. Far from it. How otherwise are we to evaluate the schemes of reformers and prophets, major and minor? Moreover the ground work of such studies must be laid well in advance, before there is any direct occasion for them; otherwise they will fail us when we do need them. There is not only an opportunity for mathematics in economics, but even a duty; and on mathematicians in an unusual degree lies the responsibility for the economic welfare of the world.

THE 21-POINT CUBIC.

By B. H. BROWN, Dartmouth College.

1. Introduction. It is the purpose of this paper to correlate certain portions of the geometry of the triangle by developing new properties of the 21-point cubic.[1] We shall give in paragraph 2 a new definition of this curve, and prove in paragraphs 2 and 3 that some 16 additional notable points lie on the curve. As the 21-point cubic is not the general circular cubic, our primary interest is not in circular cubics but in the geometry of the triangle. It is hoped that this paper will make a helpful contribution to this subject by exhibiting [2] a collection of 37 notable points (to which others may be added *ad lib.*) for which a large number of sets of 3 (4) points are readily seen to be collinear (concyclic) by means of one fundamental theorem on circular cubics.

Let us recall Neuberg's definition, and briefly enumerate his results.

Suppose that the sides opposite the vertices A_1, A_2, A_3 of a triangle have lengths a_1, a_2, a_3. Let a_4, a_5, a_6 represent the distances from any point P of the plane of the triangle to A_1, A_2, A_3 respectively. The 21-point cubic is the locus of points P for which

$$\begin{vmatrix} 1 & a_1^2 + a_4^2 & a_1^2 a_4^2 \\ 1 & a_2^2 + a_5^2 & a_2^2 a_5^2 \\ 1 & a_3^2 + a_6^2 & a_3^2 a_6^2 \end{vmatrix} = 0. \qquad (1)$$

[1] Neuberg, *Mémoire sur le tétraèdre.* Published as *Supplément V* to *Mathesis*, 1885, also *Mémoires couronnés*, etc., Belgium, 1886, pp. 1–72. See also Neuberg, *Bibliographie du triangle et du tétraèdre, Mathesis*, June, 1924, pp. 241 et seq.

[2] I am indebted to Professor R. E. Langer for the construction of the figure, and to my pupil, Mr. L. S. Kennison, for the verification of certain computations.

There is little difficulty in showing from this equation that the following points lie on the locus, together with the circular points at infinity:

(*A*) the 3 vertices A_i,
(*B*) the 3 points B_i symmetric to A_i in a_i,
(*C*) the 6 vertices C_i^+, C_i^- of the equilateral triangles constructed on a_i,
(*D*) the 3 points D_i symmetric to A_i in the lines q_{jk}, q_{kj}, where q_{jk} is the point where the side a_j meets the perpendicular bisector of a_k,
(*E*) the circumcenter E,
(*F*) the orthocenter F,
(*G*) the 2 isodynamic centers G^+, G^- such that

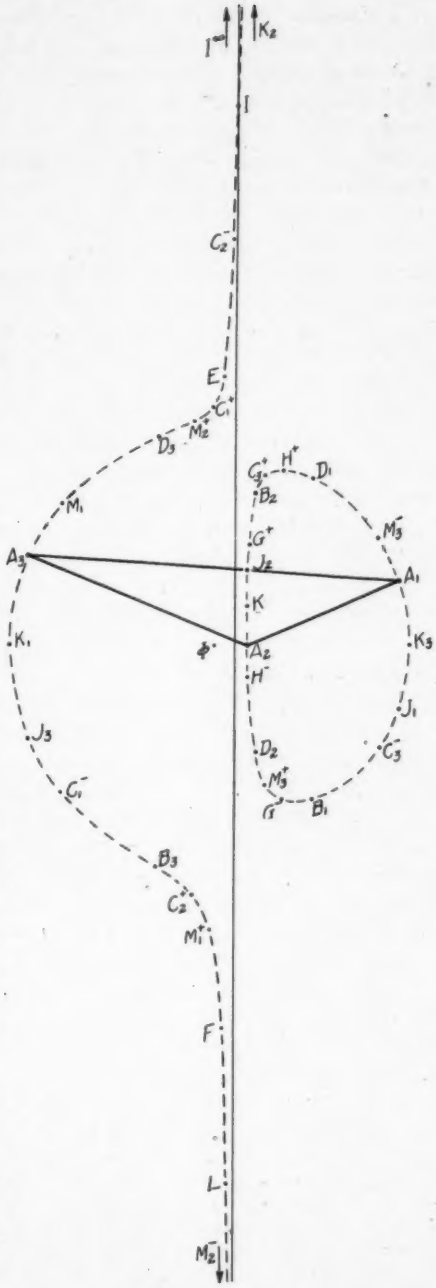
$$A_1A_2 \cdot A_3G = A_1A_3 \cdot A_2G = A_1G \cdot A_2A_3,$$

(*H*) the 2 isogonal centers H^+, H^- from which a_i subtend angles of 60° or of 120°.

The reader is referred to the figure where all these points and others found later are shown. The details of the proofs together with the genesis of the curve may be found in either of Neuberg's articles to which reference was made. The author has recently (MONTHLY, *1924*, 371–375) published a new proof that the isogonal centers lie on the cubic. After a careful search no further published results have been found.[1]

2. New definition. Let us begin by considering the following 9 points: the 3 vertices A_i of a triangle, the 4 centers K and K_i of the in- and escribed

[1] But see Neuberg, *Bibliographie*, etc., "Notre documentation est incomplète, notamment pour la cubique des 21 points."



circles, and the 2 circular points at infinity. In general, 9 points determine a cubic. But since two cubics intersect in 9 points, through any such set of 9 points will pass a pencil of cubics. The set enumerated is of this nature. To show this, we note (1) $\angle K_iA_kK = \angle KA_jK_i = \pi/2$, hence K_i, A_j, K, A_k are concyclic, and (2) K_k, A_i, K_j are collinear. Hence the cubics composed of a circle (1) and line (2) contain the 9 points and there is therefore a pencil of cubics through these points. Our discussion for the present relates to any cubic of the pencil.

We recall the theorem that if a circle meet a circular cubic in R, S, T, U and if the line RS meet the cubic in V and TU in W (similarly for RT and SU, or RU and ST), then VW is parallel to the real asymptote.[1] This theorem is fundamental for the study of the set of points with which we concern ourselves.

For any cubic in the pencil, A_j, K_i, A_k, K are concyclic, and A_jK and also K_iA_k meet the cubic in K_j, hence

THEOREM: *The tangents at K_i (K) are parallel to the real asymptote.*

Let A_iA_j cut the cubic in J_k; since K_kK cuts the cubic in A_k, A_kJ_k is parallel to the real asymptote. Let the circumcircle of the triangle cut the cubic again in I. IA_i meets the cubic in, say, A_i'. A_jA_k meets the cubic in J_i. Hence $A_i'J_i$ must be parallel to the real asymptote, that is, $A_i' = A_i$, and

THEOREM: *IA_i is tangent to the cubic at A_i.*

Consider the three circular cubics formed by the circles $J_iJ_jA_k$ and the lines A_iA_j. These pass through 8 points of our cubic, hence all four cubics have a common ninth point L. Since A_i, J_j, J_k, L are concyclic and since A_iJ_j cuts the cubic in A_k, LJ_k cuts the cubic in J_k' such that A_kJ_k' is parallel to the real asymptote. Hence $J_k' = J_k$, and

THEOREM: *LJ_i is tangent to the cubic at J_i.*

These results are true for any cubic of the pencil. Let us now confine our attention to the cubic through C_3^+, the vertex of an equilateral triangle on A_1A_2 as base. Consider the circle $A_1A_2C_3^+$. A_1A_2 cuts the cubic at J_3. Hence $C_3^+A_3$ passes through the fourth intersection of this circle and cubic. But the intersection of this circle and line is the (positive) isogonal center H^+ which accordingly is on the cubic. Next consider the circle $A_2A_3H^+$; as A_2A_3 cuts the cubic in J_1, the fourth intersection of the circle and cubic is also on A_1H^+ and is consequently a point C_1^+, vertex of an equilateral triangle on A_2A_3. Similarly C_2^+ lies on the cubic.

We now have shown that 9 points, A_i, C_i^+, H^+, and the circular points at infinity are on the cubic, and since it can easily be shown in a special case that these 9 points determine a unique cubic, they do in general, and we can identify this cubic with Neuberg's cubic. The points I, J_i, and L are not constructible until we know the direction of the real asymptote. Our principal new result is then

THEOREM: *The in- and excenters lie on the 21-point cubic.*

[1] For theorems on circular cubics mentioned in this paper see Teixeira, *Traité des courbes spéciales remarquables*, Fr. ed., 1908, vol. 1, pp. 62–83.

3. Sixteen new notable points on the 21-point cubic. Let us now discard synthetic methods for the time being and grind out a few formulas and results. Taking a triangle with vertices $(a, 0)$, $(b, 0)$, $(0, c)$, on simplifying (1) we have for the equation of the cubic

$$\{(-3ab - c^2)x + c(a+b)y\}(x^2+y^2) + 3ab(a+b)x^2 + 2c(c^2 - a^2 + ab - b^2)xy + ab(a+b)y^2 + (a^2c^2 + b^2c^2 - 3a^2b^2 + abc^2)x - c^3(a+b)y - abc^2(a+b) = 0.$$

The coördinates of the 21 notable points are given for the convenience of those who may wish to draw the cubic for other triangles. The case illustrated is for $a = 2$, $b = c = 1$. The case $a = 2$, $b = -1$, $c = 3$ is much simpler numerically but does not give quite so good a figure. A special case $a = c = 3$, $b = -1$, where the asymptote is parallel to A_1A_2, and passes through A_3, is of some interest.

(A) $(a, 0)$, $(b, 0)$, $(0, c)$;

(B) $\left(\dfrac{ab^2 - ac^2 + 2bc^2}{b^2+c^2}, \dfrac{(b-a)2bc}{b^2+c^2}\right)$, $\left(\dfrac{ba^2 - bc^2 + 2ac^2}{a^2+c^2}, \dfrac{(a-b)2ac}{a^2+c^2}\right)$, $(0, -c)$;

(C) $\left(\dfrac{b \pm \sqrt{3}c}{2}, \dfrac{c \pm \sqrt{3}b}{2}\right)$, $\left(\dfrac{a \mp \sqrt{3}c}{2}, \dfrac{c \mp \sqrt{3}a}{2}\right)$, $\left(\dfrac{a+b}{2}, \pm\dfrac{\sqrt{3}}{2}(a-b)\right)$;

(D) $\left(\dfrac{b}{a}\dfrac{2ac^2 + a^2b - bc^2}{b^2+c^2}, \dfrac{c(a-b)(ab+c^2)}{a(b^2+c^2)}\right)$, $\left(\dfrac{a}{b}\dfrac{2bc^2 + ab^2 - ac^2}{a^2+c^2}, \dfrac{c(b-a)(ab+c^2)}{b(a^2+c^2)}\right)$, $\left(\dfrac{c^2(a+b)}{ab+c^2}, \dfrac{2abc}{ab+c^2}\right)$;

(E) $\left(\dfrac{a+b}{2}, \dfrac{ab+c^2}{2c}\right)$; (F) $\left(0, -\dfrac{ab}{c}\right)$;

(G) the intersections of $B_iC_i^*$; (H) the intersections of $A_iC_i^*$.

The real asymptote is found to be

$$y - \frac{3ab + c^2}{c(a+b)}x + \frac{18a^2b^2c^2 + 9abc^4 + 9a^3b^3 + 2c^6 - 2a^2c^4 - 2b^2c^4 - 3a^3bc^2 - 3ab^3c^2}{c \cdot \Delta} = 0,$$

where

$$\Delta = c^4 + b^2c^2 + a^2c^2 + 8abc^2 + 9a^2b^2.$$

Let us call the direction of the asymptote I^∞.

The Euler line (the line EF) is

$$y - \frac{3ab + c^2}{c(a+b)}x + \frac{ab}{c} = 0,$$

so that

THEOREM: *The real asymptote of the cubic is parallel to the Euler line.*

It is easy to verify that the asymptote is the Euler line if and only if the triangle is isosceles. In this case, and only in this case, the cubic degenerates into the Euler line and a circle. If the triangle be equilateral, neither the Euler line nor the Neuberg cubic is unique.

The following curious construction suffices to fix the asymptote by giving the point where it crosses the curve. It is a known theorem that the circles $A_iq_jq_kq_j$ intersect in a point [1] φ on the circumcircle (not, in general, on the cubic). This point has coördinates

$$x = \frac{3a^3b^2 + 3a^2b^3 + a^3c^2 + b^3c^2 + a^2bc^2 + ab^2c^2 - ac^4 - bc^4}{\Delta},$$

$$y = \frac{4abc^3 + c^5 - 2a^3bc - a^2c^3 + 5a^2b^2c - 2ab^3c - b^2c^3}{\Delta};$$

and it is readily shown that

THEOREM: *The point φ is the singular focus* (that is, the intersection of the two imaginary asymptotes tangent to the cubic at the circular points at infinity).

The diametral point of φ in the circumscribed circle has the coördinates

$$x = \frac{2(c^2 + 3ab)(a+b)(c^2+ab)}{\Delta},$$

$$y = \frac{2a^2c^4 + 2b^2c^4 + 5abc^4 + 3ab^3c^2 + 3a^3bc^2 + 12a^2b^2c^2 + 9a^3b^3}{c\cdot\Delta}.$$

With a little ingenuity in combination we may readily verify the

THEOREM: *The diametral point of the singular focus φ in the circumscribed circle is the point of intersection of the tangents at A_i, that is, the point we have called I, and is also the point of intersection of the cubic and asymptote.*

The point L was defined as the intersection of the circles $A_iJ_jJ_k$. We have for it the rather unexpected

THEOREM: *L lies on the tangent to the cubic at I.*

Proof. First let us note that L, J_i, A_i, I are concyclic, for let the circle through I, A_i, J_i cut the cubic in L'. Since IA_i cuts the cubic in A_i, $L'J_i$ must cut the cubic in J_i, hence $L' = L$. Now let LI cut the cubic again at I', and consider the circle LA_iI' cutting the cubic again at J_i'. LI' cuts the cubic at I which is on the real asymptote, hence either $J_i'A_i$ passes through I, or is parallel to the real asymptote. If the former be the case, $J_i' = A_i$; if the latter, $J_i' = J_i$ and consequently $I' = I$. In the numerical example exhibited, the latter case

[1] Neuberg, *Mémoire*, etc., p. 62.

holds. We proceed to prove by continuity that this is always so. There is surely no trouble unless A_i and J_i coincide, and when we recall that J_i is on A_jA_k we see that this is impossible. Hence L is on the tangent to the cubic at I.

Eckhardt's theorem [1] states that any line through I cuts the cubic in two points equidistant from the singular focus φ. Since LI is tangent to the cubic at I, $L\varphi = I\varphi$ = diameter of the circumscribed circle. This same theorem shows that for the points A_i, the distances from φ to the cubic are extrema, since the normals at A_i pass through φ.

Finally let us define M_i^* as the intersection of $A_jC_k^*$ and a parallel through C_i^* to the real asymptote. This gives 16 new points, I, I^∞, J_i, K, K_i, L, M_i^*, and for all of these we have elementary constructions starting from the base triangle. We may add other points (less and less "notable" as the complexity of definition increases) by repetition of the use of the fundamental theorem, by the method used to obtain the point L, and by Eckhardt's theorem.

The primary purpose of this paper is to exhibit this set of 37 points $A - M$ for the convenience of those interested in the geometry of the triangle. Starting with various sets of 9 or more points, it is interesting to try to show from our fundamental theorem, perhaps with an occasional resort to analytic methods, that the remaining points lie on the cubic. In such procedures some obvious collinearities must be assumed, but a surprising number of new ones will be found. The author has noted over 75. We have shown or may show that lines through the following pairs of points contain I^∞:

$$A_iJ_i; \quad B_iD_i; \quad C_i^*M_i^*; \quad EF; \quad G^*H^*;$$

and the tangents at K and K_i. But it seems needless to catalog all the collinearities and concyclicities. Using our fundamental theorem, the student can read them more easily from the figure than from the printed page.

THE HUYGENS GOVERNOR.

By V. C. POOR, University of Michigan.

1. Introduction. The Huygens governor consists of a vertical axis, a horizontal arm and a simple pendulum attached to the end of the arm. The simple pendulum is constrained to swing in the plane of the arm and the axis. If we neglect all dissipating forces, we will have a conservative holonomic system with two degrees of freedom. The dynamical problem, then, is to study the motion of the bob subject to sets of somewhat general initial conditions. We shall also determine those positions of the bob for which the motion will be steady.

Let b and a be the length of the arm and the pendulum respectively, and let m be the mass of the bob. If we designate by ψ the angle that the pendulum makes with the arm b, and by φ the angle through which the arm swings about

[1] *Cf.* Durège, *Zeitschrift der Math. u. Phys.*, vol. 14, 1869.

the axis referred to a fixed plane of reference through the axis, then the Lagrangian function and the Lagrangian equations for the system may be written down.

The kinetic energy T and the potential energy W, if measured from the lowest point of the swing, are given by the equations

$$T = \frac{m}{2}[(b - a\cos\psi)^2\dot{\varphi}^2 + a^2\dot{\psi}^2],$$

$$W = mga(1 - \sin\psi).$$

The Lagrangian equations for a conservative system have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q},$$

where L is the Lagrangian function $T - W$. For our system there will be two such equations, one for each of the variables φ and ψ.

These equations are easily seen to be of the form

$$a^2\ddot{\psi} = (b - a\cos\psi)a\sin\psi\dot{\varphi}^2 + ga\cos\psi, \tag{1}$$

$$\frac{d}{dt}(b - a\cos\psi)^2\dot{\varphi} = 0. \tag{2}$$

The second of these equations furnishes the integral

$$(b - a\cos\psi)^2\dot{\varphi} = c_2, \tag{3}$$

which could have been obtained from the principle of *angular momentum*, since the moment of the force (mg) about the axis is zero. If we eliminate $\dot{\varphi}$ from equation (1) by the use of equation (3) and integrate (1), we find that

$$a^2\dot{\psi}^2 + (b - a\cos\psi)^2\dot{\varphi}^2 - 2ga\sin\psi = c_1, \tag{4}$$

which expresses the fact that the total energy of the system is constant. This result could have been written down from the *principle of energy* for a conservative system. A further integral from (3) and (4) may easily be obtained in the form

$$t = a\int \frac{(b - a\cos\psi)d\psi}{\sqrt{c_1(b - a\cos\psi)^2 + 2ag\sin\psi(b - a\cos\psi)^2 - c_2}},$$

which would be useful in numerical cases. Reduced to algebraic form, this integral involves the square root of an octic and gives little explicit information about the motion.

2. Oscillatory and Rotational Motions. We will now impose the following initial conditions: we will start the bob from the point $\psi = \varphi = 0$ with the angular velocities $\dot{\psi} = \dot{\psi}_0$, $\dot{\varphi} = \dot{\varphi}_0$ and leave the system to itself. We then seek to

determine the conditions under which the bob will oscillate or make a complete revolution. The above initial conditions substituted in equations (3) and (4) give

$$c_1 = a^2\psi_0^2 + (b - a)^2\dot{\varphi}_0^2, \tag{5}$$

$$c_2 = (b - a)^2\dot{\varphi}_0. \tag{6}$$

At the points where the bob comes to rest in its oscillations, ψ will be equal to zero. Equation (4) furnishes this condition in the form

$$(b - a\cos\psi)^2\dot{\varphi}^2 - 2ag\sin\psi - c_1 = 0$$

or

$$\frac{c_2^2}{(b - a\cos\psi)^2} - 2ag\sin\psi - c_1 = 0. \tag{7}$$

If we replace $b - a\cos\psi$ by r, this last equation takes the form

$$4g^2r^6 - 8g^2br^5 + [4g^2(b^2 - a^2) + c_1^2]r^4 - 2c_1c_2^2r^2 + c_2^4 = 0. \tag{8}$$

If the bob comes to rest at the topmost point, *i.e.*, if the initial velocity ψ_0 is just sufficient to carry the bob three fourths of the way around or to the point $\psi = 3\pi/2$, r will just equal b. In this case equation (8) furnishes the relation $c_1b^2 - c_2^2 = \pm 2gab^2$. Expressed in terms of the initial conditions, this becomes

$$a^2\psi_0^2 = 2a[\pm g - B\dot{\varphi}_0^2], \tag{9}$$

where

$$B = \left(1 - \frac{a}{b}\right)^2\left(b - \frac{a}{2}\right).$$

We may well distinguish three cases: $b > a/2$; $b = a/2$; and $b < a/2$.

Case I. $b > a/2, b \neq a$. In this case B is always positive, so that for $B\dot{\varphi}_0^2 < g$ there exist two values of the initial velocity $a\psi_0$, equal in magnitude but of opposite sense, which will bring the bob to rest at the point $\psi = 3\pi/2$. For a velocity numerically smaller than $a\psi_0$, the bob will oscillate and for a numerically larger velocity it will make a complete revolution. In general, if a negative g is used in determining the velocity $a\psi_0$, the velocity so determined would bring the bob to rest at the point $\psi = \pi/2$, or the lowest point in the swing. In this case evidently no such velocity exists.

If $B\dot{\varphi}_0^2 > g$, $a^2\psi_0^2$ would have to be less than a negative quantity for oscillatory motion. Since in this case there exists no initial velocity that will bring the bob to rest at the highest point, the bob will revolve for every $a\psi_0$.

For g equal to $B\dot{\varphi}_0^2$ the critical velocity is zero. There is nothing to determine which way the bob will swing, but if it starts from rest in either sense it will come to rest at the highest point, $\psi = 3\pi/2$.

Case II. $b = a$ or $b = a/2$. Here the critical velocity $a\psi_0$ is independent of the initial angular velocity $\dot{\varphi}_0$ about the axis. We have the same situation we had in Case I, but with the critical velocity $a\psi_0 = \pm\sqrt{2ag}$.

Case III. $b < a/2$. This condition makes B always negative, so that for every $\dot{\varphi}_0$ there are two equal and opposite values of $a\psi_0$ which will bring the bob to rest at the point $\psi = 3\pi/2$. For $a\psi_0$ less or greater in magnitude than the magnitude of these critical values, the bob will oscillate or make a complete revolution, respectively.

For $-B\dot{\varphi}_0^2 > g$ there are two equal and opposite values for $a\psi_0$ which will bring the bob to rest at the lowest point, $\psi = \pi/2$. For values of $a\psi_0$ numerically greater than the critical values but smaller than the critical value carrying the bob to $\psi = 3\pi/2$, the bob will oscillate through the point $\psi = \pi/2$, but for $a\psi_0$ less in magnitude the bob will oscillate but never reach the lowest point, $\psi = \pi/2$. If the bob returns to its original position in any of these cases, it will assume its initial velocities. This fact may readily be seen by replacing ψ by $2n\pi$ in equations (3) and (4), n being any positive or negative integer or zero. This shows that if the initial velocities cause the bob to oscillate or make a complete revolution, the same type of motion will persist ever after.

3. Equivalent Swings. We are also able to answer the question as to whether the in-swings and the out-swings are equal. In fact if $a \sin \psi$ be replaced by y in equation (7) and the resulting equation rationalized, we have a sextic. If the bob reaches the same height in its in-swing as in its out-swing, this sextic will have a double root. This root will appear also in its first derived equation. From this we conclude that the coefficient of y and the absolute term of our sextic must have a common factor. The ratio of these two coefficients is

$$\frac{[c_2^2 - c_1(b^2 + a^2)]^2 - 4b^2a^2c_1^2}{-4g(b^2 + a^2)[c_2^2 - c_1(b^2 + a^2)] - 16ga^2b^2c_1}.$$

After the values of c_1 and c_2 have been substituted, a careful inspection will show that there is no common factor present for general values of a, b, ψ_0 and $\dot{\varphi}_0$. The rest points are therefore at unequal distances from the horizontal plane through b.

For $\dot{\varphi}_0 = 0$ the ratio reduces to

$$\frac{(b^2 + a^2)a^4\psi_0^4 - 4a^6b^2\psi_0^4}{4g(b^2 + a^2)a^2\psi_0^2 - 16ga^4b^2\psi_0^2}.$$

We find in this case the common factor $-a^2\psi_0^2$. If in addition to this we select the factor $1/2g$ furnished by the coefficient of the highest power, we may choose $y = -(a^2\psi_0^2/2g)$ which gives exactly the rest points in the swing of a simple pendulum. In fact if we substitute the condition $\dot{\varphi}_0 = 0$ in equation (7), c_2^2 vanishes and our sextic reduces to $y = -(a^2\psi_0^2/2g)$.

Choosing b equal to zero, we have the spherical pendulum. Our ratio in this case reduces to $a^8\psi_0^4/4ga^4\psi_0^2$. The fact that equal factors appear here has no bearing on the previous argument. It is well known that the heights of the points where ψ is zero for the spherical pendulum are in general unequal. In fact

any factor we might choose would have to satisfy the cubic

$$2gy^3 + c_1y^2 - 2ga^2y + c_2^2 - c_1a^2 = 0,$$

as follows from (7) by putting b equal to zero.

The result obtained in the general case should be expected since the instantaneous centrifugal force acting on the bob is $m(b - a\cos\psi)\dot{\varphi}^2$. This force assists the out-swing and opposes the in-swing.

4. Oscillations about a State of Steady Motion. For steady motion the angular velocity $\dot{\varphi}$ must be constant. But if $\dot{\varphi}$ is constant, the quantity $(b - a\cos\psi)$, according to (3), must be constant. Therefore ψ itself must have a constant value.

Let $\dot{\varphi} = \omega$, $\psi = \alpha$, where ω is constant and α is the corresponding constant value of ψ. Substituting these values in (3), we have

$$(b - a\cos\psi)^2\dot{\varphi} = (b - a\cos\alpha)^2\omega. \tag{10}$$

If we substitute $\dot{\varphi}$ from (10) into equation (1), we get

$$a\ddot{\psi} - \frac{(b - a\cos\alpha)^4 \sin\psi\cdot\omega^2}{(b - a\cos\psi)^3} - g\cos\psi = 0. \tag{11}$$

Since ψ is constant for steady motion, $\ddot{\psi}$ must vanish. Putting ψ equal to the constant α in equation (11), the first term drops out and we have

$$\omega^2 = \frac{-g\cos\alpha}{\sin\alpha(b - a\cos\alpha)}. \tag{12}$$

If the bob be given the angular velocity ω about the axis and if its inclination to the horizontal is the corresponding α, then the motion of the system if left to itself will be steady. The bob will then move on a circle, radius $(b - a\cos\alpha)$, with the constant angular velocity ω. Suppose that we start the bob with the velocity ω about the axis but at an inclination $\alpha + \xi$ differing very little from the angle α, the defining equation (11) will then become

$$a\ddot{\xi} - \frac{\omega^2(b - a\cos\alpha)^4 \sin(\alpha + \xi)}{[b - a\cos(\alpha + \xi)]^3} - g\cos(\alpha + \xi) = 0. \tag{13}$$

We take ξ so small that the second and higher order terms are negligible; the effect on ω due to this small change in α will also be neglected in our first approximation, *i.e.*, ω will retain its constant value as defined by equation (12). If we expand those functions occurring in equation (13) into a power series in ξ and neglect all terms beyond the first degree and replace ω by its value defined by equation (12), we find that

$$a\ddot{\xi} + \frac{g[b - a\cos\alpha - 3a\cos\alpha\sin^2\alpha]}{a\sin\alpha(b/a - \cos\alpha)}\xi = 0, \tag{14}$$

which we will briefly write in the form

$$\ddot{\xi} + k^2\xi = 0. \tag{15}$$

We will designate the second factor $b - a\cos\alpha - 3a\cos\alpha\sin^2\alpha$ by v. If then we replace $\cos\alpha$ by u, we shall have the following equations:

$$v = 3au^3 - 4au + b,$$

$$k^2 = \frac{+gv}{\pm a\sqrt{1-u^2}(b/a-u)}, \tag{16}$$

$$\omega^2 = \frac{-gu}{\pm a\sqrt{1-u^2}(b/a-u)}.$$

Geometrically, the uv curve defined by the first of (16) has its maximum point at $u = -2/3$ and its minimum point at $u = 2/3$. The ordinate is always positive for $-1 < u \leqq 0$. Its minimum value at $u = 2/3$ is $v = b - 16a/9$. For b equal to a this curve cuts the positive axis at $u = (\sqrt{21} - 3)/6$ and $u = 1$. Thus for $b < a$ the curve will cut the positive axis at two points, u_1 and u_2, where $u_1 < (\sqrt{21} - 3)/6$ and $u_2 > 1$. For $a < b < 16a/9$, there will still be two intercepts such that $(\sqrt{21} - 3)/6 < u_1 < 2/3$ and $2/3 < u_2 < 1$. So for $b = 16a/9$, v will be zero and for $b > 16a/9$, v will always be positive.

It should be observed that k^2 positive implies, according to equation (15), small oscillations about a steady state. When the deviation from the steady state is slight, the motion is termed *stable motion*. For k^2 negative there will be a great deviation from the steady state, or the motion will be *unstable*. Wherever the motion is stable a steady state is possible. For real stable motion ω^2 must of course be positive.

There are several cases to consider. In the second quadrant steady motion is always possible for every value of a and b. For here $-1 < u < 0$ in equations (16) gives $v > 0$, $k^2 > 0$, $\omega^2 > 0$, or the motion is stable.

In the third quadrant, on the other hand, whatever the values of a and b, steady motion is never possible. For in this quadrant the radical will carry the negative sign, thus reversing the signs of k^2 and ω^2. The motion is thus unstable.

Results not so physically evident may be obtained from the first and fourth quadrants. Here again the radicals in the first and fourth quadrants carry opposite signs. We can thus draw immediate conclusions for the fourth quadrant as soon as we have treated the cases arising in the first quadrant, for the signs of k^2 and ω^2 will be just reversed as before. This fact justifies our conclusions for the fourth quadrant in the following table based on equations (16), the signs of v, k^2, and ω^2 being given for the first quadrant only.

5. Summary. The results of this study for the first and fourth quadrants are given in the following table from which it appears that steady motion is possible only in limited regions of these quadrants and for restricted values of a and b.

TABLE.

CONDITIONS.		CONCLUSIONS.			MOTION.	
a, b	u	v	k^2	w^2	1st Quadrant.	4th Quadrant.
$b > \frac{1}{3}a$	$0 < u < 1$	+	+	−	Unstable	Unstable
$b = \frac{1}{3}a$	$0 < u < 1$					
	$u \neq \frac{1}{3}$	+	+	−	"	"
	$u = \frac{1}{3}$	0	0	−	"	"
$a \leq b < \frac{1}{3}a$	$0 < u < u_1$	+	+	−	"	"
	$u_1 < u < u_2$	−	−	−	"	Stable
	$u_2 < u < 1$	+	+	−	"	Unstable
	$u = u_1, u_2$	0	0	−	"	"
$b < a$						
$b/a < u_1$	$0 < u < b/a$	+	+	−	Unstable	"
	$b/a < u < u_1$	+	−	+	"	"
	$u_1 < u < 1$	−	+	+	Stable	"
$u_1 < b/a < 1$	$0 < u < u_1$	+	+	−	Unstable	"
	$u_1 < u < b/a$	−	−	−	"	Stable
	$b/a < u < 1$	−	+	+	Stable	Unstable
$b < a$	$u = b/a$	−	∞	∞	Impossible	
	$u = 0$	−	$\pm$	0	Rest	
	$u = \pm 1$		∞	∞	Impossible	

THE CURVE WHOSE CURVATURE IS EVERYWHERE THE SAME AS THAT OF ITS PEDAL.

By A. R. WILLIAMS, University of California.

The locus of the foot of the perpendicular from a fixed point, or "pole," on the tangent to a curve is called the pedal of that curve.

A fairly comprehensive account of the general properties of pedal curves is given by Gino Loria in Arts. 270–274 of his *Spezielle Algebraische und Transcendente Ebene Kurven*. The German translation is by Prof. Schütte and is published by Teubner. The results given fall generally into two classes: those that are concerned with the relation between the class, order, and singularities of a given curve and those of its pedal and antipedal, and those that have reference to the area of the pedal of an oval with respect to a variable pole, or which exhibit the relation between the area of the pedal of the oval and that of the roulette traced by the pole, when the oval, carrying the pole, rolls on a fixed curve. In this connection reference may be made to the *Integral Calculus* of Joseph Edwards, Art. 422 *et seq.*, Art. 444 *et seq.*, and Arts. 665 and 672 *et seq.*

It is worth noting, though no use will be made of it in this paper, that the operation of finding the pedal is equivalent to polarization with respect to the

unit circle followed by geometric inversion with respect to the same circle. As these operations are of period two, we find the antipedal by performing them in the opposite order. With the aid of this fact and the Plücker equations the characteristics of the pedal and the antipedal of an algebraic curve may be easily deduced from those of the given curve.

1. It is the purpose of this paper to find a curve whose curvature is the same as that of its pedal at corresponding points. A direct formulation of the problem in Cartesian or polar coördinates is of course simple, but leads to an unpromising differential equation of the second order. This difficulty is avoided by expressing the curvature directly in terms of the radius vector and the perpendicular on the tangent. Thus in a system of polar coördinates, let r denote the radius vector and p the perpendicular from the origin on the tangent. Then, if ϕ and ψ are the angles which the tangent makes with the fixed line and the radius vector respectively, we have

$$p = r \sin \psi; \qquad dp = dr \sin \psi + r \cos \psi d\psi = dr \frac{rd\theta}{ds} + r \frac{dr}{ds}(d\phi - d\theta).$$

Hence

$$\frac{dp}{dr} = r \frac{d\phi}{ds} \qquad \text{or} \qquad \frac{ds}{d\phi} = r \frac{dr}{dp} = \rho,$$

where ρ is the radius of curvature. Now in figure 1 let P be a general point of a given curve, P_1 the corresponding point of the pedal with respect to the origin, and P_2 the foot of the perpendicular from the origin on the tangent to the pedal at P_1. It may be easily shown that this tangent is simply the tangent at P_1 to the circle OPP_1. Hence the triangles OPP_1 and OP_1P_2 are similar, and $OP_2 \cdot OP = OP_1^2$; or $p_1 r = p^2$, (1), where $OP_2 = p_1$, $OP_1 = p$, and $OP = r$. The angle ψ between radius vector and tangent is the same for the original curve and its pedal. OP_1, or p (which we shall call later r_1), is the radius vector of the pedal and is equal to $r \sin \psi$. The corresponding vectorial angle P_1OX we shall call θ_1. We have $\theta_1 = \theta + \psi - (\pi/2)$. These are the fundamental relations that hold between any curve and its pedal. Moreover we have found that $\rho = r(dr/dp)$ is true of any curve. Hence we have at the corresponding point on the pedal

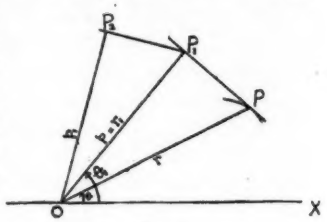


FIG. 1.

$$\rho_1 = p \frac{dp}{dp_1} = \frac{r^2}{2r - p \dfrac{dr}{dp}}$$

by means of the relation $p_1 r = p^2$. Hence the curve we seek is characterized by the relation

$$\frac{r^2}{2r - p \dfrac{dr}{dp}} = r \frac{dr}{dp}, \qquad \text{or} \qquad p\left(\frac{dr}{dp}\right)^2 - 2r \frac{dr}{dp} + r = 0. \tag{2}$$

An obvious solution, corresponding to the case of a circle, is $p = r$. The general solution is

$$4cr = (cp + 1)^2, \tag{3}$$

where c is the constant of integration. From (3)

$$r\frac{dr}{dp} = \frac{(cp+1)^3}{8c} = r\sqrt{cr} = \frac{1}{c}(cr)^{3/2}. \tag{4}$$

Now (3) is the relation between p and r on our curve. To obtain the corresponding polar equation we write

$$cp = cr\sin\psi = cr^2\frac{d\theta}{ds} = \frac{cr^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = 2\sqrt{cr} - 1,$$

whence we obtain

$$\pm\frac{dr}{d\theta} = \frac{r}{2\sqrt{cr}-1}\sqrt{c^2r^2 - (2\sqrt{cr}-1)^2} = \frac{(r\sqrt{cr}-1)}{2\sqrt{cr}-1}\sqrt{cr + 2\sqrt{cr} - 1}.$$

Separating the variables, and making the obvious substitution $\sqrt{cr} = z$, we are able to integrate. Returning to the variable r, we have

$$\pm\theta + k = 2\sin^{-1}\frac{\sqrt{cr}-1}{\sqrt{2cr}} - \sqrt{2}\log\frac{\sqrt{2cr} + \sqrt{cr + 2\sqrt{cr} - 1}}{\sqrt{cr} - 1}. \tag{5}$$

The form of the curve is certainly not apparent. We may, however, omit the constant k, which only rotates the curve about the origin, and we note that

$$2\sin^{-1}\frac{\sqrt{cr}-1}{\sqrt{2cr}} = \cos^{-1}\frac{2\sqrt{cr}-1}{cr} = \cos^{-1}\frac{p}{r} \text{ by (3);}$$

that is, the angle between p and r. Now the polar equation of the pedal is simply the relation between r_1 and θ_1 where r_1 is p and θ_1 is the angle made by r_1 (*i.e.*, p) with OX, and is equal to $\theta \pm \cos^{-1} p/r$. So we infer that the equation of the *pedal* can be got directly from (4) by replacing θ by θ_1, omitting the inverse function, and expressing r in terms of r_1 (*i.e.*, p) as given by (3). Thus we get for the pedal

$$\pm\theta = -\sqrt{2}\log\frac{\sqrt{2}(cr_1+1) + \sqrt{c^2r_1^2 + 6cr_1 + 1}}{cr_1 - 1}. \tag{6}$$

We may easily verify this. From the fundamental relation $p_1r = p^2$ which holds between any curve and its pedal, and from (3) which characterizes our curve, we have, by replacing p by r_1,

$$p_1 = \frac{4cr_1^2}{(cr_1+1)^2}, \tag{7}$$

which we may call the p and r equation of the pedal. As a check we note that

from the last relation we have on the pedal

$$\rho_1 = r_1 \frac{dr_1}{dp_1} = \frac{(cr_1+1)^3}{8c} = \frac{(cp+1)^3}{8c} = r\sqrt{cr} = \rho$$

as given by (4). That is, the pedal and the original curve have the same curvature at corresponding points. To obtain from (7) the corresponding polar equation (of the pedal) in r_1 and θ_1, we write

$$p_1 = r_1 \sin \psi_1 = r_1^2 \frac{d\theta_1}{ds_1} = \frac{r_1^2}{\sqrt{r_1^2 + \left(\frac{dr_1}{d\theta_1}\right)^2}} = \frac{4cr_1^2}{(cr_1+1)^2}.$$

Removing the factor r_1^2, and integrating as before, we obtain (6) as the polar equation of the pedal.

2. If we can determine the form of the pedal, we can of course easily pass back to the original curve. Equation (6) is at least free of an inverse trigonometric function, and fortunately a very marked further simplification is possible. From (6), using the lower sign before θ_1, we get

$$(cr_1 - 1)e^{(\theta_1/\sqrt{2})} - \sqrt{2}(cr_1+1) = \sqrt{c^2r_1^2 + 6cr_1 + 1}.$$

Squaring, and removing the factor $cr_1 - 1$, which evidently corresponds to a circle, we have

$$cr_1 = \frac{e^{\sqrt{2}\theta_1} + 2\sqrt{2}e^{(\theta_1/\sqrt{2})} + 1}{e^{\sqrt{2}\theta_1} - 2\sqrt{2}e^{(\theta_1/\sqrt{2})} + 1} = \frac{e^{(\theta_1/\sqrt{2})} + e^{(-\theta_1/\sqrt{2})} + 2\sqrt{2}}{e^{(\theta_1/\sqrt{2})} + e^{(-\theta_1/\sqrt{2})} - 2\sqrt{2}} = \frac{\cosh(\theta_1/\sqrt{2}) + \sqrt{2}}{\cosh(\theta_1/\sqrt{2}) - \sqrt{2}}. \quad (8)$$

Since this last is unchanged when we replace θ_1 by $-\theta_1$, we see, as we would expect, that we do not need to consider the double sign before θ_1 in (5). From (8) we can easily see the form of the pedal curve. As remarked it is symmetric with respect to OX. cr_1 becomes infinite when $\cosh(\theta_1/\sqrt{2}) = \sqrt{2}$, *i.e.*, when $\theta_1 = \pm 1.246$ radians. For those values of θ_1 the polar subtangent, $r_1^2(d\theta_1/dr_1)$, is finite, and equal to $\mp 4/c$. Hence the curve has asymptotes, parallel to $\theta_1 = \pm 1.246$, and at a distance $4/c$ from the origin. As θ_1 becomes infinite, cr_1 approaches 1; that is, the curve winds about the circle $cr_1 = 1$. We see from (3), (5) and (8) that the constant c is simply equivalent to a change of unit of length in r and r_1; that is, we may put $c = 1$ and still get all the curves by varying the unit of length. Taking $c = 1$ we note that when $\theta_1 = 0$, $r_1 = -5.83$. As θ_1 increases (positively),

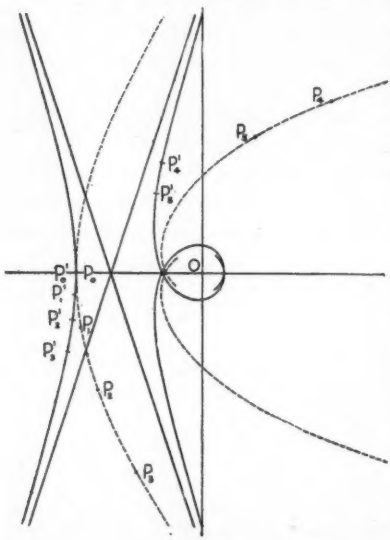


FIG. 2.

r_1 increases (negatively), becomes infinite for $\theta_1 = 1.246$, then changes sign, decreases very rapidly, and winds upon the unit circle. We have the points $(11.53, \pi/2)$, $(1.87, \pi)$, $(1.224, 3\pi/2)$, and $(1.068, 2\pi)$.

The form of the original curve, or antipedal of the pedal curve, is now easily obtained. Draw the radius vector OP_1 to any point on the pedal, and OP_2, the perpendicular from O on the tangent to the pedal at P_1. At P_1 draw the perpendicular to OP_1. Then OP and OP_2 make equal angles with OP_1 on the opposite sides of OP_1 (see Fig. 1). In our case we know that $OP = (OP_1 + 1)^2/4$, and that θ differs from θ_1 by $\cos^{-1}(OP_1/OP) = \cos^{-1}[4OP_1/(OP_1 + 1)^2]$ which rapidly approaches 0 as OP_1 (*i.e.*, r_1 or p) approaches 1. The original curve is of course symmetric with respect to OX. r becomes infinite when r_1 does in a direction perpendicular to the corresponding direction of r_1; that is, the line at infinity is twice tangent to the curve, and there are no proper asymptotes. In the graph the pedal is drawn solid; its antipedal, or the original curve, is the dotted line. $P_0'\,P_0$, $P_1'\,P_1$, $P_2'\,P_2$, etc., are pairs of corresponding points on the two curves. The original curve goes off to infinity along P_1P_2 and returns along P_3P_4. Both curves wind asymptotically around the unit circle and soon become indistinguishable. Recalling from (4) that on the original curve $\rho = r\sqrt{cr}$, and that a change of c is simply equivalent to a change of the linear unit, we may state our result in the following

THEOREM. *The curve whose curvature is the same as that of its pedal at corresponding points is a transcendental spiral, having an axis of symmetry, winding asymptotically about a circle whose center is the pole of pedalization, and whose radius of curvature, in terms of the radius of the asymptotic circle as unity, is everywhere equal to the three halves power of the radius vector drawn from the pole.*

Similarly a curve for which $\rho = (r + 1)^3/8$ has the same curvature as its *antipedal*. It is easily shown that the pedal of a logarithmic spiral is another logarithmic spiral, which, however, does not have the same curvature at corresponding points. As a check on these results we may note that it is easy, starting from the equation of the pedal curve (8), to derive the equation of the original curve from the relations $\theta_1 = \theta + \psi - (\pi/2)$ and $r = r_1 \csc \psi_1$, and show that the two have the same curvature. Also it is possible on our curve by integrating $ds = \sec\psi dr$, and using the relation $\rho = r\sqrt{cr}$, to obtain a direct relation between s and ρ, *i.e.*, an *intrinsic* equation. We can do the same for the pedal curve, but the result is not illuminating in either case.

Finally, we note that we have an illustration of the fact that two curves may be in 1 : 1 correspondence, and have the same curvature at corresponding points, without being congruent, or even similar. The ratio of ds on the one to ds on the other is variable. Thus, denoting the elements of the pedal by subscripts we have

$$ds = \frac{ds}{dr}dr = \sec\psi dr = \sec\psi_1 dr = \frac{ds_1}{dr_1}dr = \frac{dr}{dr_1}ds_1 = \sqrt{cr}ds_1 = \frac{cr_1 + 1}{2}ds_1.$$

The effect of this appears on comparing in the graph the arc lengths between pairs of corresponding points, *e.g.*, $P_1'\,P_2'$ with $P_1\,P_2$, or $P_3'\,P_4'$ with $P_3\,P_4$.

THE SOLUTION OF CERTAIN PROBLEMS IN FINANCE BY THE METHOD OF ITERATION.

By C. H. FORSYTH, Dartmouth College.

1. Introduction. The use of the method of iteration[1] in finding approximations to roots of rather complicated equations is far from new and the writer has been surprised to find no mention of it in any of the textbooks on finance. The writer has tried various methods (interpolation, binomial expansions, Newton's method, etc.) in his courses in finance, but both he and his students have consistently preferred the method of iteration. In this paper it will be applied to the determination of the investment rate in annuities and bonds in order to call the attention of readers of this journal to its peculiar fitness in such problems.

2. Application to Annuities. Since we are concerned primarily with the method we shall restrict our consideration first to the type of annuity which occurs most frequently: the one whose interval of payment coincides with the interval of conversion of interest. The equations of value to be considered resolve themselves then essentially to one of the two forms:

$$S = \frac{(1+i)^n - 1}{i} \quad \text{and} \quad A = \frac{1-(1+i)^{-n}}{i},$$

where i is the rate of interest.

Obviously neither of these equations can be solved explicitly for i. If, however, we distinguish between each of the two i's in each equation, each equation can be expressed in two ways as follows:

$$i = \frac{(1+i)^n - 1}{S}, \quad (A) \qquad i = (1 - iA)^{-1/n} - 1, \quad (A')$$

$$i = (1 + iS)^{1/n} - 1, \quad (B) \qquad i = \frac{1-(1+i)^{-n}}{A}. \quad (B')$$

Supposing now that i is the sole unknown, we can proceed to substitute a reasonably close trial value on the right-hand side to obtain a new value of i. It will be found that if equations (B) and (B') are used this new value of i will always prove to be a closer approximation to the value sought than the value tried, regardless of whether the trial value is too large or too small; and that if equations (A) and (A') are employed the values obtained will always prove divergent. If the proper equation is employed, then it merely remains to repeat or *iterate* the process—trying each new approximation in turn—until a value is obtained which reproduces itself.

[1] See Whittaker and Robinson, *Calculus of Observations*, pp. 79–94, for a general account of the method and various modifications of the method; also a good bibliography.

In iteration in general a given equation is broken up into two parts, of which one is quite frequently, but *not necessarily*, an expression of the first degree, and the process consists in substituting successive approximations alternately in the two equations but *beginning* with the one whose graph has the slope of lesser magnitude. The scheme suggested here for the solutions of annuities is equivalent to substituting first in the equation $y = f(i)$, where $f(i)$ denotes either (B) or (B'), and then (in all cases) in the equation $y = i$. In such applications then this secondary graph is always a straight line and the same straight line—the bisector of the first quadrant. It so happens, however, that the slopes of the graphs of (B) and (B') are positive throughout (as the derivative will show) but less than unity, the reverse is true in each case of the graphs of (A) and (A'). The essential forms of these graphs are shown below. There should be no difficulty in recognizing the so-called *staircase* representation of the process—convergent toward the point of intersection (or the desired root) in the case of equations (B) and (B'), and divergent in the other case.

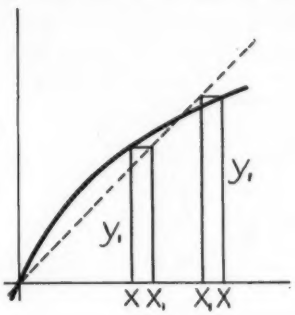


FIG. 1.

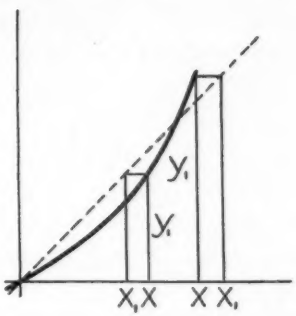


FIG. 2.

In most courses in finance the calculus would not be available but there still remain at least three methods of identifying the proper equation to be used: (1) by plotting the graph using values taken from the tables; (2) by verifying by actual application; and as a special case of the latter method, (3) by trial of a value which is both absurd and yet susceptible of mental substitution. The latter method is used rather to disqualify an equation. As an example, the substitution of the absurd value 9 for i in the right in equation (A) will obviously yield a value of iS of n digits! The same substitution in equation (A') leads to either imaginary or negative results.

3. Application to Bonds. The scheme suggested here works as well for determining the investment rate of a bond of given price but in this case we must choose between three equations and the graphs are a little more interesting or complicated—according to the point of view.

The formula for the premium or discount per dollar of the ordinary bond is

$$k = a_{\overline{n}|}(g - i) \qquad \text{(at rate } i\text{)},$$

where, in case the dividend (at rate g) and the interest (at rate i) on the investment are to be converted m times a year, we need merely to change the unit of time from a year to the conversion period (making mn such periods), and the essential form of the equation given above is unaltered.

As i appears in three separate places in the above formula, the latter may be written in either of the three forms:

$$i = g - \frac{k}{a_{\overline{n}|}} = g - \frac{ki}{1 - (1+i)^{-n}}, \tag{1}$$

$$i = \{1 - (1+i)^{-n}\}\frac{g-i}{k}, \tag{2}$$

$$i = \left\{(1+k) - \frac{gk}{g-i}\right\}^{n} - 1. \tag{3}$$

The important parts of the graphs of these equations (and of $y = i$) would appear essentially as follows:

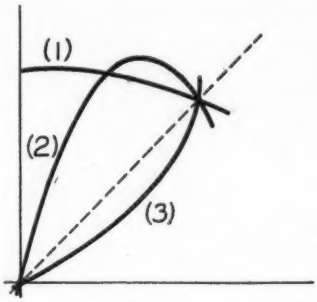


FIG. 3.

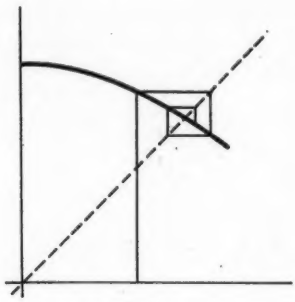


FIG. 4.

The graphs of both equations (2) and (3) have slopes which exceed unity (the slope of $y = i$). Hence these equations can be eliminated at once for purposes of iteration. Equation (1) proves to be satisfactory.

It is well known that when the slopes of the graphs of the two equations used in the process of iteration have opposite signs the staircase representation is replaced by a so-called *spiral* representation, as shown in Fig. 4, where, of course, the substitutions begin, as before, with the equation whose slope is of lesser magnitude.

The method of iteration could, no doubt, be modified to advantage wherever the derivative could be easily obtained, and in the case of annuities no great difficulty is met; but in the case of bonds such modifications lead to serious difficulties due to the necessity of taking derivatives of quotients. As a general method in finance then the writer prefers that outlined here to any other—and especially to the irksome interpolations and binomial expansions employed in current textbooks on finance. The writer's preference is influenced greatly also

by the advantages of (*a*) directness of method, (*b*) the ultimate check, and (*c*) the effective drill in the use of logarithms.

One of the most important steps in the practical application of the process of iteration is the utilization of the successive possibilities of anticipation of closer approximations. After all, the process insures merely successive closer approximations which ultimately converge to the desired value, and as we all know, such convergence is frequently exasperatingly slow. In any case, the use of an arithmetic mean of two approximations will frequently shorten the process.

We shall conclude with a simple example of an annuity to illustrate both the possibilities of shortening the process and also a suggested arrangement of the successive approximations which will help to center the attention upon the location of the desired value.

Example: Given that \$12,102 is the present value of an annuity of \$1,000 per year for twenty years; to find the corresponding rate of interest.

\$12.102 is therefore the present value of an annuity of \$1 per year for twenty years and the use of the tables shows that the rate is between 0.05 and 0.06. Employing equation (B') and using arrows to indicate substitutions, the work can be outlined as follows:

$$.05 \rightarrow .05149 \qquad .06 \rightarrow .05687$$

$$\text{Average} = .05418,$$

$$.053 \rightarrow .05322 \qquad .054 \rightarrow .05377$$

$$\text{Average} = .0535 \rightarrow .05349 \rightarrow .05349.$$

And the process could be carried on to give additional places if the table of logarithms justifies.

HENRY BRIGGS AND HIS WORK ON LOGARITHMS.

By A. J. THOMPSON, London, England.
(Communicated by KARL PEARSON, University of London.)

The year 1924 was the tercentenary of the publication of the first great table of logarithms, the *Arithmetica Logarithmica*, by Henry Briggs. It contained the logarithms, to fourteen decimal places, of all numbers from 1 to 20,000, and from 90,000 to 100,000. The Biometric Laboratory of University College, University of London, is celebrating the tercentenary by commencing the issue of *Logarithmetica Britannica*, a standard table of logarithms, which, when completed, will give the logarithms, to twenty decimal places, of all numbers up to 100,000. The first portion of this table,[1] which contains the logarithms of

[1] *Logarithmetica Britannica*, being a Standard Table of Logarithms to 20 decimal places, by Alexander John Thompson. Part IX (the first to be published), Numbers 90,000 to 100,000. Cambridge University Press, 1924. The work is intended to be completed in nine parts. Subscriptions are invited at the rate of 10 shillings per part, payable on publication. Intending subscribers should send their names to Professor Karl Pearson, F.R.S., University College, London, W. C. 1.

numbers from 90,000 to 100,000 with second and fourth differences, has already been published.

Henry Briggs was born about 1556 at Warley Wood, a hamlet near Halifax. He showed signs of mathematical ability at an early age, and proceeded to St. Johns College, Cambridge, in 1579. He took the degree of B.A. in 1581, of M.A. in 1585, and became a Fellow of his College in 1588. He remained there until 1596, when he became the first Reader in Geometry at Gresham College in London.

In 1614, Napier's first work on logarithms was published, the *Mirifici Logarithmorum Canonis Descriptio.* This book came into Briggs's hands soon afterwards. He began to read it with interest, but, by the time he had finished, his interest was changed into enthusiasm. The book was his constant companion: he carried it with him when he went abroad; he conversed about it with his friends; and he expounded it to the pupils who attended his lectures. He soon perceived that the system of logarithms which would be described, in modern language, as having 10 as base, would be much more convenient than Napier's. He described this improvement to his classes, and as soon as his courses of lectures for the session of 1614–15 were ended, he travelled to Edinburgh and discussed it (among many other subjects) with Napier. Briggs remained for a month as Napier's guest, and, on his return to London, busied himself in calculating logarithms according to the new plan. In 1616 he again visited Napier, taking with him the calculations he had made. The results of these calculations were printed in 1617, for the benefit of his personal friends, as *Logarithmorum Chilias Prima.* In this rare brochure were given the logarithms of the first 1,000 numbers, to 14 decimal places. Specimen pages are reproduced as a frontispiece to the table now being published.

In 1619, Briggs became the first Savilian Professor of Geometry at Oxford. He settled at Merton College, and resided there for the remainder of his life. He continued to carry on his computing and, in 1624, after a labor extending over a period of about eight years, produced his *Arithmetica Logarithmica.* This work contains the logarithms of 30,000 numbers, with first differences. It is accompanied by a masterly introduction, in which the construction of tables, interpolation by means of differences, and other matters of the greatest importance were dealt with for the first time. Although this work is now very rare and costly, it is said that the edition of the tabular portion was too large and that surplus copies were hawked in the streets of London at eighteenpence each. After the publication of this great book, Briggs set to work, with the help of a few friends, to fill up the large gap of 70,000 logarithms which had been left. He had almost completed them when, in 1628, Adrian Vlacq, a Dutchman, published the logarithms of the first 100,000 numbers to 10 decimal places in a book which he also called *Arithmetica Logarithmica.* Although Vlacq had only copied 30,000 logarithms from Briggs's book (cutting them down from 14 to 10 decimal places) and had calculated 70,000 himself, he described his work merely as a second edition. Briggs may have felt some disappointment at the way in

which he had been forestalled by Vlacq; he seems, however, to have cherished no resentment, but on the contrary to have felt relieved that the burden of printing 70,000 logarithms had been removed from his shoulders. In such circumstances, most men would have given up computing; but not so Briggs. At the age of 70 or thereabouts he hastened to commence another great work, the logarithms of the trigonometrical functions, and had almost completed it at the time of his death. Vlacq printed these logarithms at his own expense and published them in 1633 under the title of *Trigonometria Britannica*.

Henry Briggs died on the 26th of January, 1631. He was buried with great pomp in the chapel of Merton College, but his name only was inscribed on his tomb-stone.

No complete reprint of Briggs's great table of the logarithms of numbers has ever been made; and, up to the present year, only two 10-figure tables have been published since Vlacq produced his table in 1628. The need for an extended table has long been felt, and the present work is intended to meet this need.

No more fitting tribute than this could be paid to the memory of a great and good man on the tercentenary of his greatest work.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. GENERALIZED CONVERGENCE WITH BINARY RELATIONS.

By A. A. BENNETT, University of Texas.

The term "limit" does not of course apply to a single independent variable, but requires a correspondence between an independent and a dependent variable for it to have a meaning. The two variables do not figure in the same manner, although they enter almost analogously. Thus the notation $\lim_{x \to a} y = b$ could be written in the somewhat more symmetrical if less convenient form as $y|_{x \to a} \to b$. It might at first appear that at least the existence of the a is an essential feature of the relation. On the other hand, the expression $\lim_{n \to \infty} c_n = c$ reminds us that when the limit is a sequential one the independent variable is a natural number while the value approached by this variable in determining the limit is denoted by ∞ but is not a natural number. The formulation in this case can be so made that ∞ is never mentioned. Generalizing this and other more interesting types of limits, Prof. E. H. Moore has developed "a general theory of limits," of which a thoroughgoing discussion has been published under this title.[1] In this treatment the dependent variable is numerical throughout. For the independent variable there is introduced a binary relation of a very general sort

[1] E. H. Moore and H. L. Smith, *Am. Journ. Math.* (44), 1922 (102).

which in particular renders the existence of and even the reference to the a mentioned above unnecessary. In the paper referred to all the more important classical theorems on limits are stated and proved for this generalized definition of limit. Had the range of the dependent variable also been so generalized, some of the familiar theorems would have been rendered meaningless or incapable of expression, which is ample reason for confining attention to numerical functions. However the same process could obviously be applied for the dependent variable also. In fact the study of expressions "converging toward infinity" and of improper limits in general would suggest such a procedure. A fairly inclusive generalization is here made.

Let A be a set of elements $[a]$. Let R be a *binary* relation, that is, a relation between elements of an ordered pair selected from A. Thus given two elements, a_1 and a_2, the question is assumed to be definite and its answer determined completely by the two elements chosen in order, namely, "Does a_1 have the relation R to a_2?" In case that this relation does hold, we may use as a proposition the expression a_1Ra_2 (read "a_1 is in the R-relation to a_2"). We shall further suppose that although R applies among elements of A, it is also more extensive and there may be for example such an element as e not in A which is such that for some a, eRa.

The relation R will be required to be *transitive* in A, that is, if a_1, a_2 and a_3 are elements of A such that a_1Ra_2 and a_2Ra_3, then a_1Ra_3. Among the familiar transitive binary relations for numbers may be mentioned those denoted by $=, >, <, \geq, \leq$, while $\neq$ is not transitive.

The relation R will be required to be *antireflexive* in A, that is, it will be assumed that there is no element a of A for which aRa. This apparently stringent condition is frequently satisfied, or can be made to be satisfied by a minor modification of A or R. By a more detailed statement of the condition of convergence to be discussed this limitation could be avoided.

The relation R will be required to be *compositional*,[1] that is, for every two (not necessarily distinct) elements a_1 and a_2 of A, it is required that there exist an element a_3, of A, such that simultaneously a_3Ra_1 and a_3Ra_2. It is to be noted that while the other properties mentioned for R hold equally for the inverse relation in each case, the compositional property is unsymmetrical with respect to righthand and lefthand members.

We shall assume further the existence of a set B and of a relation S for which statements analogous to those for A and R hold throughout.

Before proceeding further we shall show that A cannot contain a fixed "limit element" a', such that for every element a, of A, $a'Ra$. For if possible suppose that such an element a' existed. On account of the compositional property (using a' twice) there must be an element of A, say a'', such that $a''Ra'$. Now a'' must be distinct from a' by the antireflexive condition, and hence by the hypothesis $a'Ra''$. By the transitive property it follows from $a'Ra''$ and $a''Ra'$

[1] This term is here used in the sense defined for the composition property in the article referred to above. The terms "binary," "transitive" and "reflexive" are familiar in logical discussions.

that $a'Ra'$, which is impossible by the antireflexive condition. However there may or may not be an element a' outside of A such that, for every a of A, $a'Ra$.

Let X be a set including A as a subset, and let x be a variable of X. We shall take x to be the independent variable in the present discussion. Similarly let Y be a set including B as a subset. Let $y(x)$ be a function of x, all of the values of $y(x)$ being in Y. The function $y(x)$ is said to *converge* with respect to the range A, and the relations R and S, if and only if the following condition [1] is satisfied: *Given any b of B, there exists an a of A, such that for each x in A for which xRa, it is true that $y(x)Sb$.* It is to be noted that $y(x)$ is not necessarily confined to B. If we insist upon inquiring as to what it is that $y(x)$ converges toward, we may answer that it converges toward a limit element d in Y but not in B, and which is such that for each b of B, dSb.

For the most familiar case of $\lim_{x \to c} y = d$, the set A is the set of all (finite) numbers (real or complex as the case may be) exclusive of c, and B is the similar set where d is excluded. Here xRa means $|x - c| < |a - c|$ and ySb means similarly that $|y - d| < |b - d|$. Since B does not include d, $|b - d|$ is greater than zero and may be called ϵ if desired, while δ may be similarly defined as the positive quantity $|a - c|$. The condition then reads in the familiar fashion: For every ϵ, > 0, there exists a δ, > 0, such that for every x, $\neq c$, for which $|x - c| < \delta$, it is true that $|y - d| < \epsilon$. The relative economy of the generalized statement may be appreciated.

That the formulation of the convergence condition in such a manner that the independent variable need not be numerical and so that the compositional property may be used in its fulness is not without significance can be observed in the problem of defining the Riemann integral. Suppose for a given interval of the real axis a one-valued real function has been defined. For each particular subdivision of the interval suppose an approximating polygon defined. We are interested in the area of the polygon as a function of the subdividing of the interval. For certain special methods of successive subdivision, the area becomes a function of a finite ordinal number indicating the particular step in the process; in the general case, however, there is no assigned sequence in the process of subdivision. If a_1 denotes one subdividing of the interval, and a_2 another, we use the fact that there is a subdividing a_3 among whose points of subdivision occur all those of a_1 and also all those of a_2, the simplest choice of this a_3 being obtained by mere superposition of the interval as subdivided in one way upon the interval as subdivided in the other.

That there is also elementary use for a non-numerical function is seen by reference to the question of uniform convergence, which we will take with respect to a scale function. Consider a set of functions $f_n(x)$ where for our purposes the independent variable is n. Assume a limit function $F(x)$ and a scale func-

[1] Only the usual fundamental convergence criterion is here discussed, and Cauchy's and other such forms will not be mentioned further except to note here that while in many of these other forms no explicit mention of the limit of the dependent variable is made, the use here to be made of the binary relations avoids explicit use of limits even in this fundamental case.

tion $s(x)$. If for every positive ϵ there is a natural number m_ϵ so large that $|f_n(x) - F(x)| < \epsilon s(x)$, for all values of x on the interval in question, whenever n exceeds m_ϵ then the set of functions is said to converge uniformly with respect to the scale function. This comes directly under our theory with the dependent variable as a correspondence and not as a number.

A discussion of further technical features and the enumeration of other examples will not be appropriate here.

II. A Method for Solving the Cubic.

By Orrin Frink, Jr., Brooklyn.

Consider the equation

$$x^3 + px + q = 0. \tag{1}$$

To complete the cube, write (1) in the form

$$x^3/8 + (3/2)xy^2 = -q/2, \tag{2}$$

where

$$y^2 = x^2/4 + p/3. \tag{3}$$

Let

$$\tfrac{3}{4}x^2y + y^3 = R. \tag{4}$$

By adding (2) and (4) and taking the cube roots,

$$x/2 + y = \sqrt[3]{-q/2 + R} \quad (1, \omega, \omega^2), \tag{5}$$

and by subtracting (2) and (4),

$$x/2 - y = \sqrt[3]{-q/2 - R} \quad (1, \omega, \omega^2). \tag{6}$$

Addition of (5) and (6) now gives the unknown x in terms of q and R; while multiplication of (5) and (6) shows, by means of (3), that $-p/3 = \sqrt[3]{q^2/4 - R^2}$, or

$$R = \sqrt{q^2/4 + p^3/27}.$$

This solution can be used conversely to extract the cube root of such an expression as $9 + 25\sqrt{-2}$ whenever the root can be exactly taken.

Thus, with a view to (6), let $x/2 - y = \sqrt[3]{9 + 25\sqrt{-2}} = \sqrt[3]{-q/2 - R}$. Here $q = -18$, $-p/3 = \sqrt[3]{9^2 - (25\sqrt{-2})^2} = \sqrt[3]{1331} = 11$.

We can now form equation (1), since we know p and q, obtaining

$$x^3 - 33x - 18 = 0.$$

We find that this has the integral root 6, so that $y = \sqrt{x^2/4 + p/3} = \sqrt{9 - 11} = \sqrt{-2}$; and $x/2 - y = 3 - \sqrt{-2}$.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Die Geburt der Modernen Mathematik. I. Analytische Geometrie. By HEINRICH WIELEITNER. Karlsruhe in Baden, G. Braun, 1924. 60 pages. Price 1 Goldmark.

This little book is No. 12 of the valuable series of monographs issued under the attractive and significant title of "Wissen und Wirken." The purpose of the series is to popularize knowledge through inexpensive booklets written by men of high standing as scholars and possessed of the happy faculty of presenting serious subjects in the everyday language that people of fair education can understand and enjoy reading. It is a series of particular value to teachers and students, and for those who read German the entire set would form a valuable nucleus for a working library.

As to Dr. Wieleitner's contribution, it should first be said that he is one of the best-known historians of mathematics in Germany at the present time. He is well equipped on the side of scholarship and is able to present his subject in a way that will appeal to the average reader who cares to know the origin of the various branches of mathematical science.

The work is divided into seven chapters relating, respectively, to (1) the birth of modern science, (2) the essential features of analytic geometry, (3) the Greek mathematics, (4) the development of algebra, (5) analytic geometry as developed by Fermat, (6) the contribution of Descartes, and (7) the general outlook.

The presentation being professedly and necessarily merely a sketch of the historical development of the subject, it is hardly necessary to speak further of the details. The reader of the monograph will be interested to learn that the first use of the term "analytic geometry" in the present sense is much later than he would expect, being first found in a work by Lacroix at the close of the 18th century. The most valuable chapter for most readers, however, will be the one on the contribution of Fermat, particularly for those who think of Descartes as the sole inventor of this branch of mathematics.

It would interest this reviewer to know Dr. Wieleitner's authority for pronouncing Oughtred's name as if it were spelled Utred. The nearest approach to historical certainty in the matter would seem to lie in the fact that some of his contemporaries spelled the name Owtred, from which we may reasonably infer the pronunciation.

It is to be hoped that similar popular series, issued at a nominal price, may soon become possible in this country.

The effect of making available for our teachers a set of booklets in paper covers, at twenty-five cents a volume, and written by those who know their subjects and can tell their stories in an interesting fashion, would be most salutary.

DAVID EUGENE SMITH.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, no. 8, October, 1924: "Invariance of the Poincaré numbers of a discrete group" by O. Veblen, 405–406; "Analytic functions and periodicity" by J. F. Ritt, 406–409; "A convergence proof for simple and multiple Fourier series" by M. G. Carman, 410–416; "On certain topics in the mathematical theory of statistics" by H. L. Rietz, 417–453.

PHILOSOPHICAL MAGAZINE, volume 48, no. 288, December, 1924: "The principle of equivalence in the theory of relativity" by T. Y. Thomas, 1056–1068.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 11, November, 1924: "Concerning sets of segments which cover a point set in the Vitali sense" by R. L. Moore, 464–466.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 9, December, 1924: "The supplementary project in mathematics" by C. A. Stone, 905–912; "A new number system" by H. C. Christofferson, 913–916; "Shall we mathematize or demathematize high school physics?" by J. M. Hughes, 916–921; "Probabilities" by J. M. Synnerdahl, 922–935; "Geometry, a laboratory science" by J. A. Nyberg, 948–953, by F. H. Sutton, 954–956, by J. O. Pyle, 956–957.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

MU THETA EPSILON OF THE UNIVERSITY OF CALIFORNIA, Berkeley, California.

Mu Theta Epsilon, women's mathematics honor society, was organized in April, 1920. The purposes of the society are: to stimulate interest in mathematics, and to encourage better work in the same; to discuss mathematical topics; and to promote congeniality among students and closer coöperation between students and professors. Members are elected by the society from the upper-division and graduate women majoring in mathematics, who have been approved by professors of the department. The professors are honorary members except that we have chosen Dr. Pauline Sperry as our one active faculty member. The average attendance during the past semester was twenty members.

The officers for the present college year are: Dorothy Godward '24, pres.; Dorothy Scott '24, vice-pres.; Veronica Satorius '24, sec.; and Mary Sweeney '24, treas.

This semester we inaugurated the plan of holding a second meeting each month, to be devoted to open discussion of a question or topic which has been announced at a previous meeting. The plan has hardly had a fair trial but promises to be a success.

The programs for the past two years and up to date are as follows:

Sept. 6, 1922. "Digit method among the ancients"—Miss Muriel Wilkinson. "Practical mathematics in surveying"—Miss Agusta Wellman.

Oct. 4, 1922. "Magic squares"—Miss Isabel Smith. The new members gave five minute speeches as follows: "Correlation of mathematics to foreign language"—Dorothy Godward. "Famous monkey problem"—Helen Growe. "Contributions of mathematics to world problems"—Florence Breed.

Nov. 1, 1922. "An interesting fourth century table"—Miss Thelma Baker. "Line coördinates"—Dr. Elsie McFarland.

Jan. 17, 1923. "The Einstein theory"—Miss Ruth van Pelt. "Time"—Miss Mary Shafer.

Feb. 7, 1923. "Applied mathematics in physics"—Mrs. Aylesworth.

Mar. 7, 1923. "Proof of the binomial theorem"—Miss Mary Scoffield. "Report of research on Huntington's first set of postulates for fields"—Miss Verna Jeffery.

April 4, 1923. "History of definitions in mathematics"—Miss Josephine Brubaker. A review of the current AMERICAN MATHEMATICAL MONTHLY—Miss Ruth Pearson.

Sept. 5, 1923. Review of the MONTHLY for April–May—Miss Igerna Hurd. "Mathematical instruction in the secondary schools"—Miss Helen Growe.

Oct. 3, 1923. "The value of mathematics in the development of the reasoning powers of the individual"—Margaret Harper. "Amicable numbers"—Edith Henderson. "Relation of mathematics to fine arts"—Gwenyth Springsteen. "An interesting problem"—Elizabeth Lange. These five minute speeches by the new members were followed by "A mathematical romance"—Dr. Elsie McFarland, "Astronomy and mathematics"—Miss Muriel Wilkinson, and "The history and origin of Mu Theta Epsilon Society"—Mrs. Aylesworth.

Nov. 7, 1923. Problems from Ball's *Mathematical Recreations*—Miss Veronica Satorius. "Theory of factoring numbers"—Miss Florence Raphael.

Feb. 1, 1924. Initiation and Banquet.

Feb. 6, 1924. "The relation of vector analysis to quaternions"—Miss Dorothy Godward.

Mar. 5, 1924. "Relation of quaternions and vectors"—Miss Ethel Barnabey.

Mar. 15, 1924. Informal entertainment for the mathematics faculty and all senior and graduate students majoring in mathematics.

April 2, 1924. "The psychology of mathematics"—Miss Beatrice Conley. "On the porism of four tangents to a twisted cubic"—Miss Vesta Sanger.

Sept. 3, 1924. "The fundamental assumptions of geometry"—Miss Margaret Harper.

Oct. 1, 1924. "Conformal mapping and its applications"—Miss Ethel Neily. "The solution of a problem in digits"—Mrs. Esther Bell. Amusing numbers on the program were: "A mathematical romance"—Dr. Elsie McFarland, and "Possibilities in teaching mathematics" —Miss Mary Shafer.

Oct. 22, 1924. Open discussion of the practical applications of mathematics.

Nov. 12, 1924. "Finite geometry"—Miss Edith Rockwell. "General displacements in space" —Miss Mary Sweeney.

(Report by Miss Neily.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3120. Proposed by V. M. SPUNAR, Chicago, Illinois.

If ρ_1 and ρ_2 be the greatest and least radii of curvature of a curved surface at a given point, φ and ψ, the angles which the normal to the surface at the given point makes with the axes of x and y, show that

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{d}{dx}(\cos\varphi)+\frac{d}{dx}(\cos\psi).$$

3121. Proposed by A. A. BENNETT, University of Texas.

Find the smallest positive integer x, not a multiple of 31, such that x^2+1 shall have two factors whose difference is divisible by 31.

3122. Proposed by A. A. BENNETT, University of Texas.

Let $P(n)/Q(n)$ denote the fraction reduced to its lowest terms which represents the minimum value of $\varphi(m)/m$, for $0 < m \leq n$, where $\varphi(m)$ is the indicator or totient of m. Show that $P(n)$ is of the form 2^a3^b for every $n \leq 2 \cdot 10^{11}$.

3123. Proposed by B. F. FINKEL, Drury College.

A circular hole, radius r, in the bottom of a flat-bottomed water-tank is covered with a weightless spherical rubber shell, radius R. Water is then poured into the tank to the depth h. What is the ratio of R to r when the shell is just on the point of rising?

3124. Proposed by M. B. PORTER, University of Texas.

$f(x)$ and $\varphi(x)$ are polynomials and all the zeros of $\varphi(x)$ are real. Let $P(x) = \varphi(x)[f^{(i)}(x)]^r$ where $f^{(i)}$ stands for the ith derivative of $f(x)$.

Prove that $[P'(0)]^2 < P''(0)P(0)$ (1) is a sufficient condition that $f(x)$ has imaginary zeros. Newton's test [1] $C_i^2 \leq C_{i-1}C_{i+1}$ where C_i is the coefficient of x^i in $f(x)$ is a special case of (1).

SOLUTIONS.

3067 [1924, 148]. Proposed by J. H. MURPHY, Pittsburgh, Pa.

On the base of a right triangle whose altitude, a, is greater than the base, b, is constructed a triangle whose vertex angle is α. What are the lengths of the two variable sides of this triangle when that part of its area outside of the right triangle is a maximum?

SOLUTION BY HARRY LANGMAN, New York City.

Let CA be the base of the triangle ABC, right-angled at C. We have given $CB = a$, $CA = b < a$. Take $AB = c$, and put $b = ea$. Hence $e < 1$.

Let P be the third vertex of the triangle to be constructed. It is evident then that the point P must be within the right angle ACB; *i.e.*, CP must intersect the hypotenuse AB—suppose in Q.

Let angle $ACP = \theta$, and $z = \cot\theta$. Then

$$AP = \frac{b\sin\theta}{\sin\alpha}, \qquad CP = \frac{b\sin(\theta+\alpha)}{\sin\alpha} = b(\cos\theta + \cot\alpha\sin\theta),$$

$$CQ = \frac{b\sin A}{\sin(A+\theta)} = \frac{b}{\cos\theta + \cot A\sin\theta}.$$

The area of that part of the triangle APC outside of the triangle ABC is $s = \frac{1}{2}AP \cdot QP\sin\alpha$,

$$\frac{2s}{b^2} = \sin\theta\left[\cos\theta + \cot\alpha\sin\theta - \frac{1}{\cos\theta + e\sin\theta}\right].$$

Hence

$$\frac{2s}{b^2} = \frac{z + \cot\alpha}{1+z^2} - \frac{1}{e+z}$$

is to be a maximum. Thus

$$-(1+z^2)^2(e+z)^2\frac{d}{dz}\left(\frac{2s}{b^2}\right) \equiv \phi(z) \equiv 2(e+\cot\alpha)z^3 + (e^2 + 4e\cot\alpha - 3)z^2 + 2e(e\cot\alpha - 1)z - (e^2+1) = 0.$$

It can be shown as follows that this equation has only one real root:

$$\phi'(z) \equiv 2\{(e+\cot\alpha)z + (e\cot\alpha - 1)\}\{3z + e\}.$$

By substitution we find that ϕ is negative for both values of z which make $\phi'(z) = 0$, *i.e.*, that both the maximum and the minimum values of ϕ are negative; hence $\phi(z) = 0$ has only one real root. Since, for values of z sufficiently large, $\phi(z)$ is positive, and, for values of z sufficiently small algebraically, $\phi(z)$ is negative, it follows that the real root of $\phi(z) = 0$ makes s a maximum. In a numerical problem this root can be found approximately by Horner's process, or its expression in terms of e and α can be found by Cardan's process.

Also solved by A. G. CLARK.

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[1] *Cf.* Netto: *Vorlesungen über Algebra*, vol. I, p. 234, edition of 1896.

3071 [1924, 206]. Proposed by E. L. POST, Cornell University.
Prove that, when $|h| < 1/e$,

$$1 + \frac{(x+h)}{1!} + \frac{(x+2h)^2}{2!} + \cdots + \frac{(x+nh)^n}{n!} + \cdots = ce^{mx},$$

where both c and m are independent of x, and the latter satisfies the equation $e^{mh} = m$. Note: Under the given conditions both members of the first equation considered as functions of x directly satisfy the mixed difference equation $f'(x) = f(x+h)$. The identification of these two equations is therefore of some interest.

SOLUTION BY OTTO DUNKEL, Washington University.

With the condition $|h| < 1/e$ the series of the problem is convergent for all values of x. For replace in it x by its absolute value X and h by its absolute value H, $H > 0$. Then in the resulting series the ratio of the $(n+2)$th term to the $(n+1)$th is

$$\left(1 + \frac{1}{n}\right)^n \frac{\left[1 + \dfrac{X}{(n+1)H}\right]^{n+1}}{\left[1 + \dfrac{X}{nH}\right]^n} H. \tag{1}$$

For a given X and H the limit of this ratio as n becomes infinite is $eH < 1$. Hence the series represents an analytic function of x; also its terms may be rearranged in any order. It will now be shown that, if we denote this function by $f(x)$ for a given h, the ratio $f(x+h)/f(x)$ is a constant m. This is obviously true if $h = 0$ and in this case $m = 1$. It will be convenient to suppose at first that $0 < h < 1/e$. If, with such a value of h, $f(x)$ vanishes, it must be for a negative value of x; and, if x_0 is the largest root, we shall suppose that $x > x_0$. Set $x = ht$, then

$$f(x) = \varphi(t) = \sum_{n=0}^{\infty} \frac{(t+n)^n h^n}{n!}, \qquad \frac{f(x+h)}{f(x)} = \frac{\varphi(t+1)}{\varphi(t)}. \tag{2}$$

For a given t the last expression in (2) may be written as a power series in h,

$$\frac{\varphi(t+1)}{\varphi(t)} = \sum_{i=0}^{\infty} \frac{a_i h^i}{i!}, \tag{3}$$

and this series is convergent for $h < 1/e$. We shall show that the a_i's do not depend on t. The equations for the determination of these coefficients are

$$(t+1+k)^k = \sum_{j=0}^{j=k} (t+j)^j \, {}_kC_j a_{k-j}, \qquad {}_kC_j = \frac{k!}{j!\,(k-j)!}. \tag{4}$$

It is readily seen that $a_0 = a_1 = 1$. Suppose then that all the a's up to and including a_k are independent of t. Then (4) is an identity in t, and on integrating both sides with respect to t, we find

$$(t+1+k)^{k+1} = A_{k+1} + \sum_{j=0}^{j=k} (t+j)^{j+1} \, {}_{k+1}C_{j+1} a_{k-j}, \tag{5}$$

where A_{k+1} is determined by setting $t = 0$. Now replace t by $t+1$ and j by $j-1$, then (5) becomes

$$(t+2+k)^{k+1} = A_{k+1} + \sum_{j=1}^{j=k+1} (t+j)^j \, {}_{k+1}C_j a_{k+1-j}. \tag{5'}$$

A comparison of this result with (4) shows that $a_{k+1} = A_{k+1}$, and hence it is independent of t. Thus by mathematical induction we see that no a will depend on t, and our theorem is proved. Therefore

$$\frac{f(x+h)}{f(x)} = \frac{f(h)}{f(0)} = \sum_{i=0}^{\infty} \frac{a_i h^i}{i!} = m. \tag{6}$$

Since $f(x)$ satisfies the equations

$$f(x+h) = mf(x), \qquad f'(x) = f(x+h), \tag{7}$$

we have $f'(x) = mf(x)$. Hence it follows that

$$f(x) = ce^{mx}, \tag{8}$$

where c is a constant. Since $f(x)$ must also satisfy the second equation in (7), we find that

$$e^{mh} = m, \tag{8'}$$

and, as h approaches zero, m must approach unity as we saw above.

An examination of the equation (8') shows that for $0 < h < 1/e$ there are two real values of m, one such that $1 < m < e$, and the other such that $m > e$. Each of these furnish a solution of the second equation of (7), but only the first gives a solution developable in the form of the series of the problem, since in the second as h approaches zero m becomes infinite. We now see that $f(x)$ does not vanish for any value of x. If h is negative, the equation (8') has one solution for m, $0 < m < 1$. It is easily seen now from (6) that the results obtained may be extended to the case of h negative provided that $|h| < 1/e$. The equation (6) or (3) gives a solution of (8') expressed in an infinite variety of forms since any real number may be substituted for x or t. The second equation of (7) has an infinite number of other real solutions, which oscillate. It is interesting to compare these results with problem **3103 [1924, 498]** and the solution of **3046 [1924, 403]**.

Also solved by the PROPOSER.

3072 [1924, 206]. Proposed by W. H. RASCHE, Virginia Polytechnic Institute.

If the four vertices, A, B, C, and D (taken in cyclic order), of a simple quadrangle are four points in a plane rigid body undergoing complanar motion, and if two opposite sides, as AB and DC, represent vectorially the accelerations of A and D respectively, then Clifford's point of the quadrangle is the center of acceleration of the rigid body.

Note: The Clifford point associated with four complanar straight lines is the point common to the circumcircles of the four triangles formed by omitting the lines in turn.

SOLUTION BY THE PROPOSER.

Definition.—Center of Acceleration. The center of acceleration of a rigid body undergoing complanar motion is a point so located in the plane of motion that—

1. The ratio of its distances from any two points of the rigid body which are in the plane of motion is equal to the ratio of the accelerations of these points.

2. The angles which the lines joining it to any two points of the rigid body which are in the plane of motion make with the directions of the accelerations of these points are equal.

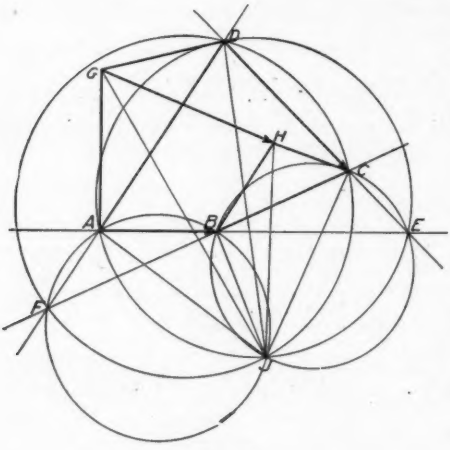

FIG. 1.

Let A and D, Fig. 1, be two points in a rigid body undergoing complanar motion, and let the vectors AB and DC represent, respectively, the accelerations of these two points. Produce the sides of the quadrangle, $ABCD$, and about each of the four triangles thus formed circumscribe a circle. Then it is to be shown that the common point, J (Clifford's point), of these four circles is the center of acceleration of the rigid body.

Proof.—I. First, consider the triangles JDC and JAB. For these triangles we have the relations: $\angle JDC = \angle JAB$, since each of these angles is measured by $\frac{1}{2}$ arc JE; $\angle BJA = \angle BFA$, because each of these angles is measured by $\frac{1}{2}$ arc AB; but $\angle BFA = \angle CJD$, for each of these angles is measured by $\frac{1}{2}$ arc DC; therefore $\angle CJD = \angle BJA$, and triangles JDC and JAB are similar.

Hence we conclude that $DC/DJ = AB/AJ$. And thus it follows that (a) the accelerations of these two special [1] points A and D make equal angles with the lines which join these points to J; and (b) that these accelerations are proportional to their distances from J.

II. Secondly, take *any* third point G in the rigid body, and on BC construct the triangle BHC similar to the triangle AGD and similarly placed. Then it is a well-established kinematic fact that the line GH represents vectorially the acceleration of point G. (See, for example, Burmester's *Lehrbuch der Kinematik*, p. 805.)

The foregoing lemma granted, it is easy to show that—

$$\text{A.} \quad \angle JGH = \angle JDC = \angle JAB,$$

$$\text{B.} \quad \frac{GH}{JG} = \frac{DC}{JD} = \frac{AB}{JA},$$

and thus that the position of point J fulfills the requirements of the definition of the center of acceleration of the rigid body.

First, consider the triangles JBC and JAD. In these triangles $\angle CJB = \angle DJA$, for $\angle CJB = \angle CJD + \angle DJB$, and $\angle DJA = \angle BJA + \angle DJB$; but $\angle CJD = \angle BJA$. Therefore, triangles JBC and JAD are similar, and hence, also, the quadrangles $JBHC$ and $JAGD$. But if these quadrangles be similar, then the triangles HJC and GJD are also similar; accordingly $CJ/HJ = DJ/GJ$, and $\angle CJH = \angle DJG$. Moreover, therefore, $\angle CJD = \angle HJG$, for $\angle CJD = \angle CJH + \angle HJD$, and $\angle HJG = \angle DJG + \angle HJD$.

Thus it follows that triangles CJD and HJG are similar; whence

$$\frac{GH}{JG} = \frac{DC}{JD} = \frac{AB}{JA},$$

and

$$\angle JGH = \angle JDC = \angle JAB, \qquad Q.\ E.\ D.$$

III. In exactly the same manner, evidently, taking the other pair of opposite sides (DA and CB) of the quadrangle as the accelerations of the two points D and C of the rigid body, it can be shown that the same point J is the center of acceleration of the body; thus the proposition as stated is true.

3078 [1924, 254]. Proposed by N. ALTSHILLER-COURT, University of Oklahoma.

Find the curve having the property that the tangent and the normal at any point determine on a fixed line two points conjugate in a given involution.

Solution by C. K. Robbins, Purdue University.

Let the fixed line be the x-axis, with the origin at the center of the involution. The involution can then be defined by $XX' = m^2$, where X and X' are coördinates of corresponding points of the involution (m may be real or pure imaginary). If the point on the curve is (x, y), then

$$X = \frac{px - y}{p}, \qquad X' = x + py \quad \left(p = \frac{dy}{dx}\right)$$

and we have at once the differential equation

$$xyp^2 + (x^2 - y^2 - m^2)p - xy = 0.$$

This may be solved by the parametric method. Let $xp - y = v$. Solve this with the differential equation for x and y, getting

$$x = \frac{p(v^2 + m^2)}{v(p^2 + 1)}; \qquad y = \frac{p^2m^2 - v^2}{v(p^2 + 1)}.$$

The fact that $p = dy/dx$ now gives a differential equation in p and v as follows:

$$\frac{vdv}{v^2 + m^2} - \frac{pdp}{p^2 + 1} = 0,$$

the integral of which is $v^2 + m^2 = c^2(p^2 + 1)$.

[1] These points are special because they are involved in the given data of the problem.

The problem reduces to eliminating v and p, using this equation and the equations expressing x and y in terms of v and p. The result is

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - m^2} = 1.$$

This is a system of confocal conics having foci at the two fixed points of the given involution. Since a conic has no singular points, the tangent and normal at points corresponding to the fixed points of the involution must coincide. Such points are necessarily imaginary. They turn out to be the four points $(\pm c^2/m, \pm [(c^2 - m^2)i]/m)$. Note that for a real conic these are imaginary whether m is real or pure imaginary.

The singular solution is $(x^2 - y^2 - m^2)^2 + 4x^2y^2 = 0$, which is a pair of conics having $(\pm m, 0)$ as their only real points in case m is real, and $(0, \pm mi)$ as their only real points in case m is pure imaginary.

Also solved by THEODORE BENNETT.

3086 [1924, 305]. Proposed by A. S. WIENER, Cornell University.
Prove that the determinant of the nth order

$$\begin{vmatrix} x_1^2 + \lambda & x_1x_2 & x_1x_3 & x_1x_4 & \cdots & x_1x_n \\ x_1x_2 & x_2^2 + \lambda & x_2x_3 & x_2x_4 & \cdots & x_2x_n \\ x_1x_3 & x_2x_3 & x_3^2 + \lambda & x_3x_4 & \cdots & x_3x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1x_n & x_2x_n & x_3x_n & x_4x_n & \cdots & x_n^2 + \lambda \end{vmatrix}$$

is divisible by λ^{n-1} and find the other factor.

SOLUTION BY J. A. BULLARD, U. S. Naval Academy.

Let A denote the matrix whose constituents are

$$(A)_{ij} = x_ix_j, \quad (i, j = 1, 2, \cdots, n).$$

Expanding the given determinant in powers of λ, we have

$$|A + \lambda| = \lambda^n + \psi_1\lambda^{n-1} + \cdots + \psi_{n-1}\lambda + \psi_n,$$

where the ψ's are functions of the x's.

If $S_1, S_2, S_3, \cdots, S_n$ denote the sums of the powers of the roots of the equation $|A - \lambda| = 0$, then

$$S_1 = \sum_{i=1}^{n} (A)_{ii} = \sum_{i=1}^{n} x_i^2,$$

$$S_2 = \sum_{i=1}^{n} (A^2)_{ii} = \sum_{i,j=1}^{n} (A)_{ij}(A)_{ji} = \sum_{i,j=1}^{n} x_i^2x_j^2 = S_1^2,$$

$$S_3 = \sum_{i=1}^{n} (A^3)_{ii} = \sum_{i,j=1}^{n} (A)_{ij}(A^2)_{ji} = \sum_{i,j,h=1}^{n} x_i^2x_j^2x_h^2 = S_1^3,$$

and similarly in general for $k = 1, 2, \cdots, n$

$$S_k = \sum_{i=1}^{n} (A^k)_{ii} = \sum_{i,j=1}^{n} (A)_{ij}(A^{k-1})_{ji} = \cdots = \sum_{h_1, h_2 \cdots h_k=1}^{n} x_{h_1}^2x_{h_2}^2 \cdots x_{h_k}^2 = S_1^k.$$

Substituting these values in Newton's formulæ connecting the sums of the powers of the roots with the coefficients, we see that $\psi_1 = S_1$, $\psi_2 = \psi_3 = \cdots = \psi_n = 0$.

Therefore,

$$|A + \lambda| = \lambda^n + \psi_1\lambda^{n-1} = \lambda^{n-1}(\lambda + S_1) = \lambda^{n-1}(\lambda + \sum_{i=1}^{n} x_i^2).$$

Also solved by THEODORE BENNETT, E. P. BUGDANOFF, E. H. CLARKE, P. J. DA CUNHA, F. HENROTEAU, J. J. NASSAU, HAZEL E. SCHOONMAKER, and the PROPOSER.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Miss BERTHA K. DUNCAN, of the University of Texas, has been appointed professor and head of the department of mathematics at Grenada College (Miss.).

Dr. G. W. HESS has been appointed professor of mathematics and head of the department at Union University, Jackson, Tenn.

Dr. W. P. OTT, of Vanderbilt University, has been appointed head of the department of mathematics at the University of Alabama.

Dr. R. L. WILDER, of the University of Texas, has been appointed assistant professor of mathematics at Ohio State University.

Mr. P. L. REA, of Marietta College, has been promoted to an assistant professorship of mathematics.

Miss INEZ MARTIN has been appointed assistant professor of mathematics at the Indiana State Normal School, Terre Haute.

Assistant Professor L. J. COMRIE, of Swarthmore College, has accepted a position in the observatory at Northwestern University.

Professor J. M. MELCHIOR, of Rockhurst College, Kansas City, Mo., has been appointed head of the department of mathematics at Campion College, Prairie du Chien, Wis.

Mr. W. H. LYONS has been appointed assistant professor of mathematics at the Kansas State Agricultural College.

Mr. G. F. ALRICH has been appointed assistant professor of mathematics at Des Moines University.

Mr. J. H. SYNNERDAHL, of Lake Forest College, has been appointed assistant professor of mathematics at Park College, Parkville, Mo.

At Washington University, St. Louis, Assistant Professor F. W. BUBB is on leave of absence for the academic year 1924–25, and is studying at the University of Chicago as a Research Council fellow. Mr. H. R. GRUMANN has been appointed instructor in mathematics and Mr. E. H. LUND instructor in applied mathematics.

At Galloway College, Searcy, Ark., Miss BERD R. ALLEN, of the Kansas State Teachers' College, has been appointed professor of mathematics and head of the department and Mrs. GEORGIA M. WILLIAMS instructor.

At Southwestern University, Georgetown, Texas, Miss VELMA TISDALE has been promoted to an assistant professorship and Mr. J. G. CHANEY has been appointed instructor.

At the University of Texas, Mr. C. A. RUPP, of Hamline University, has been appointed adjunct professor of mathematics, and Miss ELIZABETH T. STAFFORD, of Brown University, instructor.

The following appointments to instructorships have been announced:

University of Maine, Mr. E. H. HADLOCK;
Brown University, Mr. F. C. JONAH;
University of Buffalo, Mr. H. D. HIGHET;
Swarthmore College, Mr. D. B. MCLAUGHLIN;
Wofford College, Mr. W. C. HERBERT;
Athens College, Athens, Ga., Miss NANCY L. MOOREFIELD;
Mercer University, Messrs. J. L. TALLEY and C. H. BERRYMAN;
Case School of Applied Science, Mr. J. E. MERRILL;
University of Kansas, Mr. R. H. MARQUIS;
Union University, L. W. HUSSEY, E. E. STIVERT;
State College of Washington, Miss ELEANOR E. BOYD.

Professor E. L. LARKIN, director of Mt. Lowe Observatory, died October 11, 1924, at the age of seventy-seven years.

Professor P. A. Lambert, head of the department of mathematics and astronomy at Lehigh University, died on February 15, 1925.

A committee of the Society for the Promotion of Engineering Education has begun a general investigation of engineering education under a three-year grant by the Carnegie Corporation. The declared objectives of the study are (1) to clarify the educational functions and responsibilities of the colleges of engineering; (2) to establish guiding principles for the content and arrangement of curricula and the improvement of teaching; (3) to consider in what ways problems relating to engineering students, graduates and teachers may be dealt with more effectively; (4) to examine the practicability and possible benefits of closer group relationships among the colleges and with the professional organization of engineers; (5) to make an analytical comparison of the organization and practices of engineering education in Europe and America. A statement of the present status of the study is given in the *Journal of Engineering Education*, January, 1925.

Members of the Association who do not plan to keep files of the MONTHLY are urged to send back numbers of all issues to the Association library. These numbers do not have any appreciable market value when sold by a private individual to a bookseller; however the Association is able to sell complete volumes occasionally. The Association is willing to pay express charges on any numbers which members may be willing to donate. Reference to the occasional notices in the advertising columns or inquiry of the Secretary will inform any one of the issues which are specially desired.

NINTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The ninth annual meeting of the Association was held at George Washington University on Wednesday, December 31, 1924, and Thursday, January 1, 1925, following and in conjunction with the annual meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. 268 were present at the various sessions, including the following 167 members of the Association:

C. R. Adams, Brown University.
O. S. Adams, U. S. Coast and Geodetic Survey.
J. W. Alexander, Princeton University.
R. C. Archibald, Brown University.
G. N. Armstrong, Ohio Wesleyan University.
Katherine S. Arnold, Amer. Association of University Women.
C. S. Atchison, Washington and Jefferson University.
Clara L. Bacon, Goucher College.
Ida Barney, New Haven, Conn.
L. A. Bauer, Dept. of Terrestrial Magnetism.
Sarah Beall, U. S. Coast and Geodetic Survey.
W. J. Berry, Brooklyn Polytechnic Institute.
O. F. H. Bert, Washington and Jefferson College.
W. W. Bigelow, U. S. Coast and Geodetic Survey.
G. D. Birkhoff, Harvard University.
W. H. Bixby, Brig. Gen., U. S. Army, Retired, Washington.
G. A. Bliss, University of Chicago.
R. F. Borden, George Washington University.
E. W. Brown, Yale University.
H. S. Brown, Hamilton College.
R. E. Bruce, Boston University.
J. A. Bullard, U. S. Naval Academy.
W. G. Bullard, Syracuse University.
W. D. Cairns, Oberlin College.
Florian Cajori, University of California.
Minnie W. Caldwell, Chowan College.
R. D. Carmichael, University of Illinois.
E. W. Chittenden, University of Iowa.
G. R. Clements, U. S. Naval Academy.
Abraham Cohen, Johns Hopkins University.
J. L. Coolidge, Harvard University.
C. H. Currier, Brown University.
J. E. Davis, Drexel Institute.
W. W. Denton, University of Michigan.
Alexander Dillingham, U. S. Naval Academy.
C. N. Dickinson, Hollins College.
Arnold Dresden, University of Wisconsin.
J. A. Eiesland, West Virginia University.
Harry English, High Schools, Washington.
J. T. Erwin, George Washington University.
H. S. Everett, Bucknell University.
Fay Farnum, Graduate School, Cornell University.
B. F. Finkel, Drury College.
W. B. Fite, Columbia University.
J. A. Foberg, Pennsylvania State Dept. of Public Instruction.
T. M. Focke, Case School of Applied Science.
J. G. Fowlkes, University of Wisconsin.
J. F. Fox, U. S. Bureau of Standards.
T. C. Fry, Western Electric Co., New York.
W. V. N. Garretson, Rutgers College.
Leslie J. Gaylord, Agnes Scott College.
H. M. Gehman, University of Pennsylvania.
F. J. Gerst, St. Louis University.
O. E. Glenn, University of Pennsylvania.
Alice A. Grant, Oldsfield School, Glencoe, Md.
T. E. Gravatt, Pennsylvania State College.
W. M. Hamilton, U. S. Nautical Almanac Office.
Harris Hancock, University of Cincinnati.
J. G. Hardy, Williams College.
Isabel Harris, University of Richmond.
G. A. Harter, Delaware College.
W. E. Heal, U. S. Coast and Geodetic Survey.
E. R. Hedrick, Southern Branch, University of California.
P. E. Hemke, Langley Aeronautical Laboratory, Langley Field, Va.
Archibald Henderson, University of North Carolina.
Robert Henderson, Equitable Life Assur. Co., New York.
C. L. Herron, Hillsdale College.
A. O. Hickson, Brown University.
T. H. Hildebrandt, University of Michigan.
C. E. Hille, Princeton University.
A. A. Himwich, New York.
H. L. Hodgkins, George Washington University.

JEWELL C. HUGHES, University of Arkansas.
L. S. HULBURT, Johns Hopkins University.
I. W. HUNTZBERGER, Central High School, Washington.
W. A. HURWITZ, Cornell University.
NELLE L. INGELS, Interstate Commerce Commission.
R. P. JOHNSON, Carnegie Institute of Technology.
F. E. JOHNSTON, University of Illinois.
L. S. JOHNSTON, Pennsylvania State College.
L. C. KARPINSKI, University of Michigan.
A. E. KENNELLY, Harvard University.
J. H. KINDLE, University of Cincinnati.
J. R. KLINE, University of Pennsylvania.
E. H. KOCH, Jr., Brooklyn Technical High School.
W. D. LAMBERT, U. S. Coast & Geodetic Survey.
A. E. LANDRY, Catholic University of America.
C. G. LATIMER, Tulane University.
SOLOMON LEFSCHETZ, University of Kansas.
ANNA D. LEWIS, Lake Erie College.
FLORENCE P. LEWIS, Goucher College.
ZUNG-NYI LOH, Graduate School, Cornell University.
A. C. LUNN, University of Chicago.
W. D. MACMILLAN, University of Chicago.
E. S. MANSON, Ohio State University.
SISTER MARIOLA, Rosary College.
L. C. MATHEWSON, Dartmouth College.
J. F. MESSICK, Emory University.
BESSIE I. MILLER, Rockford College.
J. A. MILLER, Swarthmore College.
G. R. MIRICK, Lincoln School.
C. N. MOORE, University of Cincinnati.
R. L. MOORE, University of Texas.
C. C. MORRIS, Ohio State University.
MARSTON MORSE, Cornell University.
F. R. MOULTON, University of Chicago.
FRANK MORLEY, Johns Hopkins University.
PAUL MUEHLMAN, Loyola University.
F. D. MURNAGHAN, Johns Hopkins University.
F. H. MURRAY, Dalhousie University.
J. R. MUSSELMAN, Johns Hopkins University.
C. A. NELSON, Johns Hopkins University.
G. D. OLDS, Amherst College.
MARGARET C. PACKER, Hood College.
ANNA J. PELL, Bryn Mawr College.
F. W. PERKINS, Harvard University.
L. C. PLANT, Michigan Agricultural College.
O. J. RAMLER, Catholic University.
W. W. RANKIN, JR., Agnes Scott College.
ARTHUR RANUM, Cornell University.
C. H. RAWLINS, U. S. Naval Academy.
C. J. REES, University of Delaware.
B. L. REMICK, Kansas State Agricultural College.
J. N. RICE, Catholic University.
R. G. D. RICHARDSON, Brown University.
H. L. RIETZ, University of Iowa.
J. F. RITT, Columbia University.
H. M. ROBERT, JR., U. S. Naval Academy.
G. EDNA ROBINSON, Shorter College.
E. D. ROE, JR., Syracuse University.
W. H. ROEVER, Washington University.
R. E. ROOT, U. S. Naval Academy.
G. A. ROSS, Central High School, Washington.
F. H. SAFFORD, University of Pennsylvania.
J. B. SCARBOROUGH, U. S. Naval Academy.
HAZEL E. SCHOONMAKER, Western College for Women.
JOSEPH SEIDLIN, Alfred University.
W. F. SHENTON, U. S. Naval Academy.
L. P. SICELOFF, Columbia University.
LUDWIK SILBERSTEIN, Eastman Kodak Co.
L. L. SILVERMAN, Dartmouth College.
T. MCN. SIMPSON, Randolph-Macon College.
MARY EMILY SINCLAIR, Oberlin College.
H. E. SLAUGHT, University of Chicago.
CLARA E. SMITH, Wellesley College.
D. E. SMITH, Columbia University.
SARAH E. SMITH, Mount Holyoke College.
W. M. SMITH, Lafayette College.
J. T. SPANN, University of Maryland.
T. H. TALIAFERRO, University of Maryland.
J. S. TAYLOR, University of Pittsburgh.
R. W. THOMAS, Washington & Jefferson College.
W. A. TITSWORTH, Alfred University.
BIRD M. TURNER, West Virginia University.
H. W. TYLER, Massachusetts Institute of Technology.
H. S. UHLER, Yale University.
C. E. VAN ORSTRAND, U. S. Geological Survey.
OSWALD VEBLEN, Princeton University.
W. J. WALLIS, Central High School, Washington.
J. L. WALSH, Harvard University.
J. M. WEST, Pennsylvania State College.
WILLIS WHITED, Pennsylvania State Dept. of Highways.
A. H. WILSON, Haverford College.
ELIZABETH W. WILSON, Central High School, Washington.
D. W. WOODARD, Howard University.
E. W. WOOLARD, U. S. Weather Bureau.
J. W. YOUNG, Dartmouth College.

The meetings of the American Association for the Advancement of Science began with the formal opening Monday evening at Memorial Continental Hall where Dr. C. D. WALCOTT, secretary of the Smithsonian Institution, gave the

address of the retiring president on "Science and Service" (*Science*, January 2, 1925) and the Honorable CHARLES E. HUGHES, Secretary of State, spoke on "Some Aspects of International Coöperation" (*Science*, January 9, 1925), saying that scientists can and do aid the promotion of peace greatly by developing new and enlarged conceptions of national and international interests, by thinking in terms of peoples and not simply of governments, by urging insistently that international coöperation in scientific research is not only desirable but absolutely necessary. This meeting was followed by a reception given at the new National Museum under the auspices of the Washington members of the American Association. The President of the United States addressed the visiting scientists at the White House on Wednesday noon (*Science*, January 9, 1925), expressing the great dependence which the government places on science, paying tribute to the scientists as the wonder workers of all the ages, and recognizing that they are animated by a profound purpose to better the estate of men. Among the other general sessions were a public lecture on Thursday afternoon by Professor A. E. DOUGLAS, director of the Observatory of the University of Arizona, on the University's eclipse expedition of September, 1923, and a showing on Thursday evening of motion pictures taken on the western trip that followed the Toronto meeting of the British Association, by Doctor E. E. SLOSSON, director of Science Service.

Professor M. I. PUPIN of Columbia University was elected president of the American Association for 1925, Dr. W. J. HUMPHREYS of the U. S. Weather Bureau was elected general secretary, and Professors E. B. WILSON, B. M. DUGGAR and VERNON KELLOGG were elected members of the Executive Committee. On nomination by the section committee for Section A, Professor W. H. ROEVER was chosen vice-president, Professor R. C. ARCHIBALD, secretary, and Professor E. W. CHITTENDEN member of the section committee.

A large exhibition of scientific apparatus, books, products and research methods was held, chiefly in the gymnasium of George Washington University, in easy access from the lecture halls where many of the science meetings were held. These exhibits included, among others, the publications of the American Mathematical Society, many science texts by various publishers, Professor W. H. Roever's mechanism for illustrating systems of lines of force, a display of graphical and other material by the University of Missouri chapter of Pi Mu Epsilon, extensive collections of instruments and maps from the U. S. Coast and Geodetic Survey, the Bureau of Standards, the Metric Association, the U. S. Department of Agriculture and various manufacturers of scientific apparatus. A new and very novel exhibit was the latest development of the representation of the earth's surface as adopted by the Corps of Engineers, U. S. Army, obtained by building a model from the topographic map to show the hills and valleys in miniature form, photographing the illuminated model so as to obtain relief in pictorial form, the lighting producing a deepness of shade graded according to the steepness of slopes, and finally "overprinting" this relief upon the topographic map. Professor W. H. Roever was in charge of the mathematical portion of the exhibition.

On Tuesday evening the mathematicians and visiting scientists were entertained at a reception and dance given at the Washington Club by the Columbian Women of George Washington University; Miss Elizabeth W. Wilson, a member of the Mathematical Association, is president of the Columbian Women. Numerous facilities were afforded by George Washington University under the chairmanship of Dean H. L. Hodgkins, as well as by the local committees, for the comfort and convenience of its guests and this was recognized in a vote of thanks which was offered at the closing session by Professor H. S. Everett and adopted heartily.

The joint dinner of the mathematicians with 125 persons present was held on Thursday evening at Franklin Square Hotel which was the headquarters of this group. After a witty introduction by the toastmaster, Professor Coolidge, speeches were made by Professor Cajori, who spoke of the solid worth and reward which came to those who attended these meetings even in the face of various difficulties; by Professor Hedrick, who explained the continuing need of printing American mathematical journals in Germany because of the large increase in printing costs since the war and urged strongly the necessity of giving American mathematics its proper place in the world through feasible facilities for printing books prepared by our first-rank mathematicians, citing as a notable forward step the example of Mrs. Carus in providing so generously for the publication of the Carus Mathematical Monographs, the first of which is just off the press; by Professor Moore who urged that the teachers of mathematics and our mathematical societies should be less reluctant to maintain before the world the importance of mathematics; and by President Olds who, a teacher for forty-five years, told of the glorious work of the teacher, the great history of mathematics in this country and the fine spirit of unity exemplified by such gatherings as this.

The American Mathematical Society held its thirty-first annual meeting with sessions for the reading of papers Monday afternoon and Tuesday morning and afternoon. Fifty-four papers were read at these sessions. The second Bôcher prize was divided equally between Professor E. T. Bell of the University of Washington for his memoir "Arithmetical Paraphrases" (*Transactions*, Volume 22) and Professor Solomon Lefschetz for his memoir "On certain numerical invariants of algebraic varieties with applications to Abelian varieties" (*Transactions*, Volume 22). The second Josiah Willard Gibbs lecture under the auspices of the Society was delivered by Mr. Robert Henderson, Vice-President of the Equitable Life Assurance Society, on Tuesday evening at the auditorium of the Department of the Interior Building. The title of the lecture was "Life insurance as a social service and as a mathematical problem." It will be printed in the *Bulletin* of the Society.

The mathematical fraternity, Pi Mu Epsilon, met for a business session and dinner on Monday with delegates from the chapters in various colleges and universities.

The History of Science Society met in conjunction with Section L on Wednesday and Thursday. Professor L. C. Karpinski, Vice-President of Section L,

presided at the session, and among the other papers were those by Professor Cajori on "Leibniz, the master builder of mathematical notations," and by Professor Archibald on "Benjamin Peirce." Because of these two papers the Thursday session of the Association was postponed until a later hour of the afternoon.

The sessions of the Mathematical Association consisted of joint sessions on Wednesday and Thursday mornings and separate sessions on the two afternoons. The program was prepared under the following Committee on Program: F. D. Murnaghan, chairman; A. A. Bennett and T. McN. Simpson. Abstracts of most of the papers are given, numbered in accordance with the numbers of the papers.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION.

(1) "Remarks on the foundations of geometry" by Professor OSWALD VEBLEN, Princeton University, retiring president of the Society.

(2) "The foundations of the theory of algebraic numbers" by Professor HARRIS HANCOCK, University of Cincinnati, retiring vice-president of Section A.

1. Professor Veblen's address will appear in the *Bulletin* of the American Mathematical Society.

2. Professor Hancock's address was printed in *Science* for January 2 and 9, 1925. After making some general observations on the ultimate recourse to mathematics in the formulation of physical theories, he introduces by simple examples the notion of number realms. He then points out that in the extension of those realms, by the introduction of new numbers, it becomes necessary to make certain modifications in order that the usual theorems of arithmetic hold in the more general realms; examples were given illustrating this point. The ideal factors of Kummer and Dedekind were then mentioned and other points in the theory illustrated.

Professor J. C. Fields of the University of Toronto, chairman of Section A, who was prevented by illness from attending the meeting, sent a telegram of greetings; in his absence President Rietz of the Mathematical Association presided at this session.

FIRST SESSION OF THE ASSOCIATION.

(3) "Outlines of the fields of research: The mathematics of economics" by Professor G. C. EVANS, Rice Institute.

(4) "Outlines of the fields of research: General analysis" by Professor T. H. HILDEBRANDT, University of Michigan.

(5) "On the empirical representation of certain production curves" by C. E. VAN ORSTRAND, geophysicist in the United States Geological Survey.

(6) "Preliminary report of the Committee on Standard Departments of Mathematics" by Professor R. D. CARMICHAEL, University of Illinois, Chairman.

3. Professor Evans's paper appeared in full in the MONTHLY for March.

4. Professor Hildebrandt's paper, which considered general analysis from the standpoints of the theory of abstract sets and of functionals in a particular class or the function field, as exemplified by the work of Fréchet and E. H. Moore, respectively, and which gave suggestions as to tasks that are worth doing, or at least attempting, in these fields, will appear in an early issue of the MONTHLY. In the discussion which followed, Professor Chittenden expressed the judgment, contrary to a suggestion made in the paper, that the idea of neighborhood cannot well be dispensed with as a fundamental notion.

5. Mr. Van Orstrand's paper appeared in the *Journal of the Washington Academy of Sciences* for January 19, 1925. This abstract, like the address, is given with the permission of the Director of the Geological Survey.

Production data representing the yearly output from individual mines or from groups of mines included within areas of the order of magnitude of states or nations can generally be represented by a curve which has a zero value at the origin of time—the curve rises irregularly and oftentimes quite abruptly to a maximum value and then declines rather slowly, presumably to a zero value, in an interval of time which may be assumed to be finite, or infinite.

Ten functions satisfying the mathematical conditions were defined by means of theorems of the theory of functions. The list includes the seven types of the frequency distribution deduced by Pearson and three additional functions, the most important of which is the equation,

$$y = ax^m e^{-bx^n}.$$

The value, $n = 1$, does not minimize the sum of the squares of the residuals, but it is a sufficiently close approximation to the true value for most practical purposes. The remaining constants are easily evaluated by the method of least squares. The observation equations are written in the logarithmic form, and the theoretical weight, y^2, is assigned to each equation. An approximate evaluation of the constants of Pearson's types I and III was made, using the method of moments, but this method, in its present state of development, is not easily applied to an incomplete set of observations.

6. The report made by Professor Carmichael for the committee appointed at the Cincinnati meeting was purely preliminary. He stated that it appeared quite difficult to determine just what should be treated in such a report. Thus, as an example, when it was proposed to draw up as a standard a graded series of the number of hours per week which should be taught by college teachers of mathematics, judgments of extreme approbation and of extreme mistrust as to the wisdom of the Association's presuming to set a standard were expressed. A view was supported by some members of the committee that a very extended study should be made of the questions given to the committee, and the chairman added that it now appeared that any brief analysis such as at first contemplated would be insufficient to enable the committee to reach a definite formulation.

Professor Cairns remarked that the very fact that such divergent views were held made more evident the importance for the American colleges of a full study

of the questions and that it is very desirable that the committee should through the Association's financial backing be enabled to meet for face-to-face consideration and debate on the general questions of what constitute a good college department of mathematics. Professor Berry said that many institutions emphasize strong instructional work and that such cases should be distinguished from the large institutions which take as their distinctive work emphasis on research. Professor Armstrong desired information as to how certain courses should be grouped, *e.g.*, the teaching of mathematics in the department of mathematics or of education, surveying in the department of mathematics or of physics.

JOINT SESSION OF THE ASSOCIATION WITH SECTIONS A, B AND D OF THE AMERICAN ASSOCIATION AND THE MATHEMATICAL SOCIETY.

(7) "Stellar evolution" by Professor H. N. RUSSELL, Princeton University, representing the Astronomical Society.

(8) "Is the universe finite?" by Professor ARCHIBALD HENDERSON, University of North Carolina, representing the Mathematical Association.

An audience of 250 was present at this joint session. It was presided over by Professor W. F. G. Swann of Yale University, who was relieved later by Professor J. A. Miller of Swarthmore College.

7. Professor Russell gave a classification of stars according to brightness and color. He pointed out how the plotting of the surface temperatures against the amount of light emitted gives definite clusterings on the diagram, running in what Eddington calls the main sequence from hot white stars to cooler red stars but with a branch composed of the giant stars for which brighter light is accompanied by lower temperature. It was explained how the consideration of inner temperature, rate of radiation of heat, and radio-active changes, with the convertibility of mass and energy, are at present used to account for the evolution of the stars which is implied in the diagram above referred to.

8. Professor Henderson's paper will appear in full in the MONTHLY.

The discussion was shared in by Dr. Silberstein and Professors Veblen, MacMillan and Russell.

SECOND SESSION OF THE ASSOCIATION.

(9) "Application of Ritz's method to practical problems in engineering" by WILLIS WHITED, Pennsylvania State Department of Highways.

(10) "Browse: A course in scientific literature" by Professor BESSIE I. MILLER, Rockford College.

(11) "New conformal world maps derived from elliptic functions" by Dr. O. S. ADAMS, United States Coast and Geodetic Survey.

9. In 1908 Walter Ritz published in *Crelle's Journal*, vol. 135, page 1, a method whereby a large number of practical problems in elasticity can be solved with ample accuracy for all practical purposes and with little difficulty. Many of these problems involve partial differential equations of a high order, the solution

of which is hardly practicable except in a few special cases. This method does not seem to be understood in this country so widely as its value would justify. The present paper was written to explain and illustrate the method by determining the stresses in a concrete roadway paving slab, a subject which is of vast importance to American engineers at the present time.

The essence of this method, as applied to determining the stresses in a body subjected to flexure, consists essentially of formulating an expression for the elastic deflection of the body with unknown coefficients. The equations of internal and of external work must then be deduced. By Castigliano's principle of least work, these must reduce to minima. These equations of work are then differentiated with respect to the various unknown coefficients and their values determined by substituting in the following equations: Let the equation of internal work be $L_1 = F(a_1, a_2, \cdots a_n, x', y', z')$ and that of external work be $L_0 = G(a_1, a_2, \cdots a_n, x', y', z')$, then $L_1 = L_0$,

$$\frac{dF}{da_1} \cdot \frac{dG}{da_2} = \frac{dF}{da_2} \cdot \frac{dG}{da_1},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{dF}{da_1} \cdot \frac{dG}{da_n} = \frac{dF}{da_n} \cdot \frac{dG}{da_1}.$$

These equations will be sufficient to determine the values of the unknown coefficients $a_1, a_2, \cdots a_n$, from which the stresses can be determined by the ordinary principles of elastics.

In the case of the flat plate of uniform thickness and isotropic material, the internal work of flexure is

$$L_1 = \frac{mEh^3}{24(m+1)} \int\int \left\{ \frac{m}{m-1} \left[\left(\frac{d^2z}{dx^2} \right)^2 + \left(\frac{d^2z}{dy^2} \right)^2 \right] + \frac{2}{m-1} \frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} + 2 \left(\frac{d^2z}{dxdy} \right)^2 \right\} dxdy,$$

in which m = Poisson's ratio = 3, E = modulus of elasticity of the plate = 2,000,000, h = thickness of the plate = 6″, assuming a load of one pound and all dimensions in inches.

The problem of the flat roadway slab of indefinite extent resting on an elastic subgrade and loaded in the middle is comparable to that of a sheet of ice resting on water. This problem was solved by Hertz in 1884 (*Wiedemann's Annalen*, vol. 22, p. 449) and a more useful solution is given by Föppl in *Vorlesungen über Technische Mechanik*, vol. 5, p. 103 ff., 1922.

If the load is applied at the edge of the slab, we can place

$$z = Ce^{-a(x+y)} \cos ay \, (\sin ax + \cos ax),$$

the origin being at the loaded point. The work of compression of the soil,

considered elastic, is $90z^2$ per square inch, the coefficient of elasticity of the soil being assumed as 180. Treating these expressions as indicated above, we obtain $a = .03217$, $C = .00000511$, and the extreme fiber stress in the slab—longitudinally under the load—equals .0714 lb. Placing $x = y = 0$ gives $z = C$, whence the external work $= \frac{1}{2}C + \int 90z^2 dxdy$.

If the slab is warped up at the edge, as it frequently is on a cool night, so that the edges are not supported by the soil, we can place

$$z = Ce^{-ay}(\sin ay + \cos ay)\left(1 - \frac{3x}{2l} + \frac{x^3}{2l^3}\right),$$

in which l = width of the unsupported portion; and, proceeding as before, we find $C = .00000000396l^2$ and $a = \dfrac{1.1646}{l}$, the extreme fiber stress, longitudinally under the load, = .0724 lb. per sq. inch, and the extreme fiber stress, transversely at the line of support opposite the loaded point, = .08014 lb. per sq. inch. These stresses are independent of the width of the overhanging portion of the slab, but, of course, do not include the stress due to the weight of the slab itself, nor that due to a neighboring wheel load. These results may seem absurd, but a little reflection will show that the greater the width of the overhanging portion, the wider the distribution of the stress.

If the slab is warped up in the middle, as is frequently the case on a warm day, so that it is supported only at the edges, we can place

$$z = Ce^{-ay}(\sin ay + \cos ay)\left(1 - \frac{3x^2}{2l^2} + \frac{x^3}{2l^3}\right),$$

in which $2l$ = unsupported width, and proceeding as before we find $a = .9473/l$, $C = .000000001573l^2$ and the extreme transverse fiber stress—under the load, which is in the middle of the length as well as of the width of the slab—equals .03787 and the longitudinal fiber stress at the same point = .02867 lb. per sq. inch, both independent of the width of the unsupported portion.

If the slab is warped up in the center, and loaded at the end of the warped-up portion, *e.g.*, at an expansion joint or a transverse crack, we can place

$$z = Ce^{-ay}\cos ay\left(1 - \frac{3x^2}{2l^2} + \frac{x^3}{2l^3}\right)$$

and, proceeding as before, we find $C = .0000000053l^2$ and $a = .8563/l$ and the extreme fiber stress transversely under the load = .1076 lb. per sq. inch, also independent of the width of the warped-up portion.

The soil supporting the corner of a slab may be removed by erosion or settlement, or it may be warped up as mentioned above. In this case a one-pound load on the corner will produce an extreme fiber stress of $3/h^2$ at all points of the unsupported portion of the slab. If the slab is 6″ thick, the stress will be = .0833.

If, however, the slab is supported by an elastic subgrade, the stresses can be obtained by the solution of the following differential equation:

$$\frac{d^4y}{dx^4}+\frac{2}{x}\frac{d^3y}{dx^3}+\frac{12E_0}{Eh^3}y=0,$$

in which E_0 = modulus of elasticity of the subgrade and x = the distance from the corner, measured in the line bisecting the corner angle which is 90°. If we place $12E_0/Eh^3 = k$, the solution of the equation is as follows:

$$\begin{aligned} y = {} & A\left(1-\frac{kx^4}{2\cdot 3^2\cdot 4}+\frac{k^2x^8}{2\cdot 3^2\cdot 4\cdot 6\cdot 7^2\cdot 8}-\cdots\right) \\ & +(B+B'\log_e x)\left(x-\frac{kx^5}{3\cdot 4^2\cdot 5}+\frac{k^2x^9}{3\cdot 4^2\cdot 5\cdot 7\cdot 8^2\cdot 9}-\cdots\right) \\ & +B'\left[x+\frac{kx^5}{3\cdot 4^2\cdot 5}\left(-1+\frac{1}{3}+\frac{2}{4}+\frac{1}{5}\right)\right. \\ & \left.-\frac{k^2x^9}{3\cdot 4^2\cdot 5\cdot 7\cdot 8^2\cdot 9}\left(-1+\frac{1}{3}+\frac{2}{4}+\frac{1}{5}+\frac{1}{7}+\frac{2}{8}+\frac{1}{9}\right)+\cdots\right] \\ & +C\left(x^2-\frac{kx^6}{4\cdot 5^2\cdot 6}+\frac{k^2x^{10}}{4\cdot 5^2\cdot 6\cdot 8\cdot 9^2\cdot 10}-\cdots\right). \end{aligned}$$

This is a harmonic function and will give the same stresses adjacent to the corner as those just given, but they will diminish as x increases, becoming alternately positive and negative, approaching zero as an asymptote.

Professor Murnaghan reported that a discussion of Ritz's method is to be found in a recent number of the *Bulletin* of the French mathematical society.

10. The paper called "Browse" gave the results of a course in scientific literature. Most of the students enrolled were among the best in college and the results obtained were therefore of primary interest in connection with the teaching of "best" students.

11. This paper gave a short review of the work heretofore done on world maps and then showed a series of fifteen slides that illustrated both what had already been done and what advances have recently been made by the author. In all nine new projections were shown, four being for a single hemisphere and five for the complete sphere. Seven of the nine new projections belong to the rhombic class, since they are associated with a rhombus which may become a square.

It is a somewhat remarkable fact that of the eleven projections of this kind that have been computed and constructed for geographic purposes, ten of them have been produced by the U. S. Coast and Geodetic Survey. The first projection of this class was computed and constructed by C. S. Peirce, at that time an assistant in the Coast and Geodetic Survey, and published by him in the Superintendent's Report for 1877. This projection which Peirce called the quincuncial projection, together with the nine new examples shown in connection with this paper, makes up the number contributed by the Coast and Geodetic Survey.

The most important projections shown for practical purposes are the three rhombic projections of the western hemisphere including the one in the square; the rhombic projection of the whole world in a regular hexagon; that of the whole world in the 60°, 120° rhombus; the whole world in a six-pointed star; and finally, the whole world within an ellipse. Some of these maps compare very favorably with other maps in use, many of which are not conformal and consequently such that the meridians and parallels often do not intersect at right angles. The complete theory and discussion of the various projections will be given in a forthcoming publication of the U. S. Coast and Geodetic Survey.

Meeting of the Board of Trustees of the Association.

Nine members of the Board were present at the two sessions.

The following fifty-five persons and three institutions were elected to membership on applications duly certified:

To Individual Membership.

C. S. Allen, M.S. in E.E. (Lafayette). Asst. Prof., Physics, Muhlenberg Coll., Allentown, Pa.
E. E. Allen, A.M. (Park Coll.). Prof., Occidental Coll., Los Angeles, Calif.
C. F. Barr, M.S. (Chicago). Instr., Purdue Univ., W. LaFayette, Ind.
May M. Beenken, Ed.B. (So. Branch, California). Asso., So. Branch, Univ. of California, Los Angeles, Calif.
R. L. Cary, A.M. (Haverford). Woodbrook, Govans, Baltimore, Md.
Mae E. Conn, A.M. (So. California). Instr., Univ. of Southern California, Los Angeles, Calif.
Hollis Cooley, A.M. (Middlebury). Teacher, High School, Hackensack, N. J.
Jessie B. Edmondson, A.B. (Wilson). Teacher, Western High School, Washington, D. C.
Carolyn Eisele, A.B. (Hunter). Instr., Hunter Coll., New York, N. Y.
G. G. Entz, B.S. in Archit. Eng. (Columbia). 1352 Poinsettia Pl., Hollywood, Calif.
Bess M. Eversull, Ph.D. (Cincinnati). Instr., Smith Coll., Northampton, Mass.
Irving Fisher, Ph.D. (Yale). Prof., Political Econ., Yale Univ., New Haven, Conn.
S. L. Fitch, 512 W. Sullivan St., Olean, N. Y.
F. C. Hall, A.M. (Columbia). Teacher, Union High School, Redondo, Calif.
F. C. Hartwick, Licentiate in Math. and Sc. (Fribourg, Switz.). Prof., Univ. of Dayton, Dayton, Ohio.
C. M. Hobart, A.M. (Illinois). Prof., Northern Ill. State Teachers Coll., De Kalb, Ill.
G. B. Horlocker, A.B. (South Dakota). Instr., High School, Huron, S. Dak.
G. K. Howe, B.S. (Worcester Poly. Inst.). Prof., Atlanta Univ., Atlanta, Ga.
R. S. Hoyt, M.S. (Princeton). Engr., Dept. of Development and Research, Amer. Tel. and Tel. Co., New York, N. Y.
I. W. Huntzberger, Ph.D. (American Univ.). Instr., Central High School, Washington, D. C.
Olive M. Jones, A.M. (Columbia). Head of Dept., Queens Coll., Charlotte, N. C.
P. C. Jones, B.S. (Mass. Inst. of Tech.). Elec. Engr., Goodyear Tire and Rubber Co., Akron, Ohio.
Dora E. Kearney, A.M. (Minnesota). Supervisor, Teaching of Math. in Jr. and Sr. High School, State Teachers Coll., Valley City, S. Dak.
B. F. Kimball, A.M. (Harvard). Instr., Cornell Univ., Ithaca, N. Y.
P. A. Knedler, A.B. (Muhlenberg). Instr., High School, Cranford, N. J.
Martha M. Knepper, A.M. (Missouri). Prof., State Teachers Coll., Cape Girardeau, Mo.
E. J. Larkin, B.S. (Cooper Union). Instr., Cooper Union, New York, N. Y.
Sophia H. Levy, Ph.D. (California). Asst. Prof., Univ. of California, Berkeley, Calif.
Viola Lindberg, A.B. (Bethany). Instr., Bethany Coll., Lindsborg, Kans.

DECA LODWICK, Ph.B. (Iowa). Head of Dept., High School, Long Beach, Calif.
(Miss) ZUNG-NYI LOH, A.B. (Wellesley). Grad. Student, Cornell Univ., Ithaca, N. Y.
ADA A. MCCLELLAN, B.S. (Chicago). Teacher, High School, Long Beach, Calif.
MARY E. MARRS, Grad. N. Tex. Normal Coll. Asst. Prof., John Tarleton Agric. Coll., Stephenville, Tex.
W. E. MASON, M.S.E. (Michigan). Instr., Appl. Math., So. Branch, Univ. of Calif., Los Angeles, Calif.
E. B. MORRIS, Ph.B. (Sheffield Sc. School). Actuary, Life Dept., The Travelers Ins. Co., Hartford, Conn.
ANNA M. MULLIKIN, Ph.D. (Penna.). Instr., Germantown High School, Germantown, Pa.
W. C. MYERS, A.M. (American Univ.). Head Teacher, McKinley Tech. High School, D. C. *Riverdale, Md.*
HARRY OKEAN, B.S. (Alfred Univ.). Instr., Physics and Alg., High School, Lambertville, N. J.
B. C. PATTERSON, A.B. (Washington & Jefferson). Grad. Student and Student Asst., Johns Hopkins Univ., Baltimore, Md.
A. W. PHILIPS, A.M. (Chicago). Head of Dept., State Teachers Coll., Emporia, Kans.
BORIS PODOLSKY, B.S. in E.E. (So. California). 715 W. Court St., Los Angeles, Calif.
G. T. PUGH, Ph.D. (Vanderbilt). Head of Dept., Winthrop Coll., Rock Hill, S. C.
PATRICK QUILTY, C.E. Instr., Cooper Union, New York, N. Y.
G. EDNA ROBINSON, A.M. (Missouri). Head of Dept., Shorter Coll., Rome, Ga.
S. A. ROWLAND, A.B. (Ouachita). Grad. Student, Univ. of Chicago, Chicago, Ill.
RAFAEL SANCHEZ, B.S. (Coll. of A. & M. Arts, Porto Rico). Instr., High School., Mayaguez, P. R.
J. H. SCHAD, A.M. (Johns Hopkins). Instr., Arts and Sc. Coll., Univ. of Maryland, College Park, Md.
Y. S. SETO, B.S. (Huron Coll., S. Dak.). Grad. Student, Columbia Univ., New York, N. Y.
J. T. SPANN, B.S. (Mississippi). Asst. Prof., Univ. of Maryland, College Park, Md.
T. H. TALIAFERRO, Ph.D. (Johns Hopkins). Prof., Univ. of Maryland, College Park, Md.
Sister MARY THECLA, Ph.D. (Fordham). Teacher, St. Agnes-Sem., Brooklyn, N. Y.
G. S. VAN FLEET, B.S. (Montana State Coll.). Instr., Univ. of Michigan, Ann Arbor, Mich.
E. E. WALDEN, A.B. (Hendrix Coll.). Prof., Lambuth Coll., Jackson, Tenn.
W. J. WALLIS, A.M. (Columbia Univ.), LL.B. (George Washington). Head of Dept., Washington High School; Teacher, George Washington Univ., Washington, D. C.
R. M. WINGER, Ph.D. (Johns Hopkins). Asso. Prof., Univ. of Wash., Seattle, Washington.

To Institutional Membership.

POMONA COLLEGE, Claremont, Calif.
UNIVERSITY OF WESTERN ONTARIO, London, Ontario, Canada.
UPSALA COLLEGE, East Orange, N. J.

The following have been appointed associate editors of the MONTHLY for the year 1925:

N. H. ANNING	H. J. ETTLINGER	C. N. MILLS
R. W. BURGESS	B. F. FINKEL	F. D. MURNAGHAN
W. B. CARVER	TOMLINSON FORT	D. E. SMITH
OTTO DUNKEL	C. F. GUMMER	V. H. WELLS.

The Secretary and Dean T. M. FOCKE of Case School of Applied Science were re-appointed as the representatives of the Association on the Council of the American Association for 1925. It was voted to hold the annual meetings in December 1925 and 1926 at Kansas City, Mo., and Philadelphia, Pa., respectively, in affiliation with the American Association; it was felt by the Trustees that the general program of the American Association meetings and the reduced railroad rates obtainable through this large gathering will attract a large number

of our members in the far West to the Kansas City meetings. The Trustees approved the organization of a Southern California Section, the sixteenth section of the Association, in response to a request of about forty members who were present at a meeting in Los Angeles, November 22, 1924. By the instructions of the Trustees the Secretary cabled the greetings of the Association to the Third Pan-American Scientific Congress then in session at Lima, Peru; the Association's representatives to this Congress were Professors E. V. HUNTINGTON and D. N. LEHMER.

Recognizing that the function of the National Committee on Mathematical Requirements had been fulfilled, this committee was formally discharged at the request of the chairman of the committee; it will be recalled that appropriate recognition of the work of the committee was made at the Cincinnati meeting as follows:

> "The Mathematical Association of America in session at Cincinnati desires to place on record its sense of high appreciation of the splendid piece of work brought to happy conclusion by the National Committee on Mathematical Requirements. The devotion displayed by the various members of the Committee in articulating and organizing the thought of the country on a wide range of important problems has made possible the accomplishment of a task which might well have been deemed of formidable magnitude.
>
> "Especially does the Association desire to recognize the great ability, tact, and energy with which the undertaking has been prosecuted by the Chairman of the Committee, JOHN WESLEY YOUNG, who has to a notable degree placed the stamp of his own personality on one of the greatest achievements of coöperative effort in the history of mathematical teaching."

It was voted to express appreciation of the strenuous and able efforts put forth by the Joint Committee on Membership of the Society and the Association and especially by Professor CLARA E. SMITH who has been actively in charge of that work. Letters have been sent to 3,500 teachers of mathematics in the colleges and universities of the United States and Canada, both members and non-members, appealing for additional memberships. On the Association side, over fifty have become members through this agency, and beside this immediate result there has been a useful campaign of educating the rank and file of our teachers to the importance of enlivening alliances with active societies like these. It is a startling fact that over half of our teachers, including many in junior colleges and the stronger normal schools, belong to neither organization; here is missionary work close at hand for all of our members!

A formal report for the Carus Publication Committee was made by Professor Slaught, who presented a proposed working agreement concurred in by Mrs. Carus, whereby the Open Court Publishing Company undertakes full responsibility for the publication and distribution of the Monographs. This plan involves a very liberal increase in expense above that contemplated in the original deed of gift and to this Mrs. Carus has acceded with the most cordial good will.

The Trustees by unanimous action adopted the recommendations of the

Committee ratifying the agreement and directed the Secretary to send to Mrs. Carus their hearty thanks for her generous spirit and to express to her their deep appreciation of her interest in the Association and its endeavor to serve the cause of mathematics through this monograph series.

An arrangement is being made whereby Chancellor A. B. CHACE of Brown University will publish under the auspices of the Association his scholarly work on the Ahmes Papyrus. It is to be in two volumes. The first will contain the free translation, commentary and notes; the second will contain over one hundred large plates on which may be traced the correspondence of the hieratic text with the hieroglyphic translation, with the English transliteration, and with the literal English translation. The terms of issue will be unusually favorable to the Association and to its members. The Trustees expressed their strong appreciation of Dr. Chace's generosity, and appointed as a committee to act for the Association, Professor R. C. ARCHIBALD, chairman, and Professor DAVID EUGENE SMITH. Detailed announcement will be made later in the year.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The Secretary-Treasurer announced the names of those elected to membership. He reported also the death of the following members:

C. B. AUSTIN, Professor of mathematics, Ohio Wesleyan University (September 9, 1924).

J. E. HODGSON, Professor of mathematics, West Virginia University (April 11, 1924).

JOSEPH LIPKA, Assistant professor of mathematics, Massachusetts Institute of Technology (January 15, 1924).

C. S. SPERRY, Professor of engineering mathematics, University of Colorado (August 10, 1924).

R. E. WILSON, Dean of men, Northwestern University (December 30, 1923).

R. S. WOODWARD, former president of the Carnegie Institution (June 29, 1924).

The election of officers for the year 1925 was conducted by mail and in person at this meeting, the tellers, Professors Messick and Simpson, reporting the result of the balloting as follows:

For President: J. L. Coolidge, 188 votes; J. W. Young, 177 votes.

For Vice-Presidents: A. A. Bennett, 196 votes; Elizabeth B. Cowley, 164 votes; Dunham Jackson, 228 votes; R. E. Moritz, 136 votes.

For additional members of the Board of Trustees (to serve until January 1928): R. C. Archibald, 249 votes; L. P. Eisenhart, 195 votes; E. V. Huntington, 199 votes; D. N. Lehmer, 166 votes; G. A. Miller, 179 votes; C. N. Moore, 133 votes; F. D. Murnaghan, 98 votes; H. L. Rietz, 228 votes.

The following were accordingly declared elected:

President: J. L. COOLIDGE, Harvard University.

Vice-Presidents: A. A. BENNETT, University of Texas; DUNHAM JACKSON, University of Minnesota.

Additional members of the Board of Trustees: R. C. ARCHIBALD, Brown University; L. P. EISENHART, Princeton University; E. V. HUNTINGTON, Harvard University; H. L. RIETZ, University of Iowa.

Professor Slaught made a report for the Committee on the Carus Monographs and a hearty vote of thanks was given to Mrs. Carus for her continued and increased generosity.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 16, 1924.

RECEIPTS.			EXPENDITURES.		
Balance Dec. 4, 1923		$5,527.56	Publishers' bills (Nov. '23–Sept. '24)		$5,448.28
1923 indiv. dues	$ 432.76		Insurance on reserve copies of MONTHLY		6.40
1923 instit. dues	9.50		Manager's office		23.38
1923 subscriptions	10.00		Editor-in-chief's office		633.36
1924 indiv. dues	5,707.85		Committee on Membership		70.50
1924 instit. dues	629.20		Joint Committee on Membership		91.95
1924 subscriptions	833.09		Part expense Register		108.40
Contributions to 1924 expenses	13.00		Secretary-Treasurer's office:		
Initiation fees	357.00		Postage	$272.40	
Life membership fees	144.52		Bond	5.00	
Sale copies MONTHLY	71.54		Safety deposit	4.00	
Advertising	630.00		Office supplies	24.95	
For Monograph	3.75		Express, tel., etc.	37.10	
For *Annals* subscription	1.50		Clerical work	793.95	
Interest Oberlin Savgs. Bank	106.87		Printing	374.05	
Interest Peoples Bkg. Co.	72.59		Cincinnati meeting	50.00	
Interest Treasury Note	32.81		Paid copies of MONTHLY	1.75	
Interest Liberty Loan	21.25		Refund on subscriptions	33.05	
Interest Hardy Fund	120.00		Library expense	58.79	
			Paid for *Annals* subscription	1.50	
Total 1924 receipts		9,197.33			1,656.54
			Annals subvention		250.00
Total assets to the end of 1924 business		$14,724.79	Paid to sections from initiation fees		156.50
			Paid to B. F. Finkel int. Hardy Fund		120.00
Total expenditures		8,565.31	Total expenditures		$8,565.31
			Cash on hand		39.17
			Checking account		1,142.61
			Oberlin Savings Bk. account		2,098.53
Balance to the end of 1924 business		6,159.48	Peoples Bkg. Co. account		1,567.04
			Liberty Bond		500.00
			U. S. Treasury Note		500.00
Received on 1925–26 business		687.87	Treas. Sav. Cert.		1,000.00
Book balance Dec. 16, 1924		$6,847.35	Bank balance Dec. 16, 1924		$6,847.35

Of the funds on hand, $213.36 is held in a separate Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1925, and $1,000 is held, as the beginning of a permanent endowment fund, in the form of U. S. Treasury Savings Certificates, the present value being estimated at $1,035.

Aside from the above-mentioned funds on hand, the first payment of $1,200 on the Carus Fund amounts with the interest accruing at 4 per cent. compounded quarterly to $1,250.01.

When the accounts were closed December 16, 1924, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1924 the following items:

Bills Receivable.		Bills Payable.	
1924 individual dues	$200.00	(Either paid in December or estimated.)	
Advertising	150.00	Publishers' bills (Oct.–Dec. 1924)	$2,100.00
	$350.00	Register	650.00
		Manager's office	25.00
		Editor-in-chief's office	80.00
		Other editors' postage	20.00
		Committee on Membership (Joint campaign)	150.00
		Secretary-Treasurer's office	80.00
		Initiation fees due to sections	300.00
		Printing annual ballots, programs, etc.	200.00
		Life Membership Fund	213.00
			$4,018.00

If to the balance on 1924 business shown in this report, $6,159.48, there be added the bills receivable, $350, and there be subtracted the estimated amount of bills payable, $4,018, there results an estimated final balance on 1924 business of approximately $2,500. The corresponding estimated final balance one year ago on 1923 business was $2,400; the Association continues thus to have a narrow margin of profit.

W. D. Cairns, *Secretary-Treasurer.*

ANNUAL MEETING OF THE TEXAS SECTION.

The fourth annual meeting of the Texas section of the Mathematical Association of America was held in San Antonio, Texas, November 28, 1924. In the morning the section held a joint session with the mathematical section of the Texas State Teachers Association. These meetings were held in the primary room of the First Presbyterian Church.

In the morning Professor J. M. Bledsoe of the East Texas State Teachers College, Commerce, Texas, presided as Chairman of the mathematical section of the Texas State Teachers Association. The afternoon session was presided over by Professor L. R. Ford of Rice Institute, Houston, Texas, Chairman of the Texas section of the Association. There were seventy-five (75) present, including the following members of the Association:

W. L. Ayres, A. A. Bennett, H. E. Bray, J. E. Burnam, E. Dice, H. J. Ettlinger, L. R. Ford, H. Halperin, A. S. Hathaway, G. P. Horton, G. K. Mayne, C. A. Rupp, G. S. Smith, G. R. West, W. M. Whyburn, C. N. Wunder.

The following papers were presented at the joint meeting in the morning:

(1) "Junior high school mathematics" by J. F. HOWARD, San Antonio Junior High Schools.

(2) "The viewpoint governing mathematical instruction" by Miss ELIZABETH DICE, North Dallas High School.

(3) "A theorem concerning parallelograms" by J. E. BURNAM, Simmons College.

(4) "Economic value of the study of mathematics" by H. J. ETTLINGER, University of Texas.

(5) "Reorganization of secondary education in the light of survey findings" by F. J. KELLY, University of Minnesota.

(6) "Why young women should include one year of mathematics in their college training" by Miss EDNA GRAHAM, West Texas State Teachers College.

(7) "The beauties of geometry" by J. W. CALHOUN, University of Texas.

In the afternoon the following papers were presented:

(1) "The solution of equations by successive approximation" by L. R. FORD, Rice Institute.

A study of a very general method of solving numerical equations of all kinds; particular reference was made to quadratic equations in illustrating the method.

(2) "Darboux integrals" by G. P. HORTON, University of Texas.

It was shown in this paper that the Riemann theory of integration follows from a study of the relation of the Darboux integrals to the Lebesgue integral of the maximum function.

(3) "Some properties of the annulus" by J. E. BURNAM, Simmons College.

(4) "Geometrography" by A. A. BENNETT, University of Texas.

Geometrography is the science of analyzing geometrical constructions into elementary steps, classifying and weighting these elementary steps, and thus weighting geometrical constructions.

(5) "Anharmonic ratios" by A. S. HATHAWAY, Boerne, Texas.

(6) "Fourier series" by H. J. ETTLINGER, University of Texas.

This paper dealt with the convergence of a Fourier expansion of a general class of functions and the Gibbs phenomenon of the series thus obtained.

(7) "Statistical correlation and linear independence" by H. E. BRAY, Rice Institute.

The purpose of the paper was to show how the formula for multiple correlation can be derived by means of a useful theorem on determinants.

(8) "Approximation formulæ" by A. J. MARIA, Rice Institute (by invitation).

This paper dealt with a method of constructing simple formulæ to replace complicated ones; the derivation of a simple formula for the length of an ellipse is used to exemplify the method.

(9) "On explosion transformations" by C. A. RUPP, University of Texas.

The paper dealt with questions attendant upon the formation of envelopes by replacing the points of a curve by circles whose radii vary in accordance with some known law.

(10) "On an extension of a definition of a Green's function" by W. M. WHYBURN, Texas A. and M. College.

This paper appeared in full in the Sept.–Dec. (1924) number of the *Annals of Mathematics*, 2d series, vol. 26, pp. 125–130.

The following officers were elected to serve for the next year: A. A. BENNETT, University of Texas, chairman; W. M. WHYBURN, Texas A. and M. College, vice-chairman. The term of the present Secretary-Treasurer continues for three (3) more years.

A very pleasant feature of the meeting was a dinner in honor of the Texas section at Our Lady of the Lake College, a first rank senior college for girls, conducted by the Sisters of Divine Providence. Father Constantineau, President of the college, welcomed the guests in behalf of the Sisters of Divine Providence, who were the hosts of the occasion. Dr. A. A. Bennett, Chairman elect, responded for the section and expressed the appreciation of the section for the hospitality extended to them. Dr. L. R. Ford, retiring chairman, also made a few remarks.

H. J. ETTLINGER, *Secretary-Treasurer.*

AN ALGEBRAICALLY REDUCIBLE SOLUTION OF THE CUBIC EQUATION.[1]

By GLENN JAMES, Southern Branch, University of California.

The essential steps in this treatment of the cubic equation are deriving a fundamental theorem on interpolation, isolating a root and solving for this root by means of the interpolation theorem. The irreducible case is given separate treatment, partly to introduce the general theory but mainly because the separate treatment gives formulæ for the roots that are simpler than those immediately deducible from the general solution.

1. **Interpolation Theorem:** *If $f(x)$ is a polynomial such that*

$$0 < f(x_1)/[f(x_1) - f(x_n)] \leqq 1 \tag{A}$$

and

$$\lim_{x_n \to r} f(x_1)/[f(x_1) - f(x_n)] = 1, \tag{B}$$

then r is a root of the equation

$$f(x) = 0. \tag{C}$$

PROOF: The first hypothesis implies that $f(x_1)$ and $f(x_n)$ have different signs for the denominator must have the same sign as the numerator. Hence, clearing of fractions and subtracting $f(x_1)$ from both members of the inequality gives[2]

$$-f(x_n) \leqq 0 \text{ whenever } f(x_1) > 0$$

[1] Presented to the American Mathematical Society, February 28th, 1925, under the title, "A Complete Solution of the Cubic Equation."

[2] Assumption (*A*) excludes the trivial case in which $f(0) = 0$.

and

$$-f(x_n) \gtreqless 0 \text{ whenever } f(x_1) < 0.$$

Thus a root of (c) is either x_n or lies in the interval bounded by x_1 and x_n. Hypothesis (B) implies that

$$\lim_{x_n \to r} [f(x_1) - f(x_n)] = f(x_1),$$

from which it follows that

$$\lim_{x_n \to r} f(x_n) = 0.$$

Whence,[1]

$$f(r) = 0.$$

2. Solution of the Irreducible Cubic Equation. Consider the cubic equation in the form

$$x^3 + px + q = 0, \tag{1}$$

where

$$q^2/4 + p^3/27 \gtreqless 0, \qquad q \neq 0 \tag{2}$$

and the signs of the roots have been changed, if necessary, to make the constant term positive. The condition (2) can be written in the form

$$|q| \gtreqless 2\sqrt{-p^3/27} = (2|p|/3)\sqrt{-p/3} - e^2, \tag{3}$$

where e^2 depends upon p and q. Substituting this form of q in the left member of (1), giving x the value $\sqrt{-p/3}$ and simplifying, we get $-e^2$. Hence one root, at least, lies in the closed interval bounded by 0 and $\sqrt{-p/3}$. Now divide[2] the roots of (1) by $\sqrt{-p/3}$, that is, substitute $x = x'\sqrt{-p/3}$. Whence (1) takes the form

$$x^3 - 3x + c = 0, \tag{4}$$

where

$$0 < c = |q|\sqrt{27/-p^3} \gtreqless 2, \tag{5}$$

the second inequality resulting from (2).

Let r_1, r_2, r_3 be the roots of (4), written in ascending order of magnitude. Then from the above transformation and from consideration of $f(-1)$, $f(-2)$, $f(1)$ and $f(2)$ we see that[3]

$$-1 \gtreqless r_1 \gtreqless -2, \qquad 1 \gtreqless r_2 > 0, \qquad 2 \gtreqless r_3 \gtreqless 1.$$

[1] The restriction that $f(x)$ be a polynomial is more stringent than necessary to make this step valid. It would suffice to assume that $f(r)$ is defined.

Throughout this paper the notion of limit approaching shall include, "vacuously," the case in which the limit is actually reached.

[2] Since the roots are real p cannot be zero.

[3] Smaller intervals can be found, for instance $\sqrt[3]{c/2} \leqq r_2 > 0$, but the evaluation of this radical makes more labor in the evaluation of the root than does the use of a larger interval.

Solution for r_2. Consider the linear interpolation formula

$$x_{n+1}/x_n = f(0)/[f(0) - f(x_n)]. \tag{6}$$

Letting $f(x) = x^3 - 3x + c$, this becomes

$$x_{n+1}/x_n = c/[c - (x_n^3 - 3x_n + c)] = c/[3x_n - x_n^2], \tag{7}$$

whence [1]

$$x_{n+1} = c/[3 - x_n^2]. \tag{8}$$

The hypotheses of the interpolation theorem will be satisfied by (6) when we have proved that the terms of the sequence $x_1, x_2, \ldots x_n \ldots$ never increase and are positive, for then

$$0 < f(0)/[f(0) - f(x_{n+1})] \leqq 1$$

and x_n will approach a limit, different from zero, by 8; it follows that

$$\lim_{n\to\infty} x_n/x_{n-1} = \lim_{n\to r_2} f(0)/[f(0) - f(x_n)] = 1.$$

By (8) we have

$$x_2 = c/[3 - x_1^2] = c/[3 - 1] = c/2, \tag{9}$$

whence

$$|x_2| \leqq |x_1| = 1. \tag{10}$$

Now

$$|c/[3 - x_{n+1}^2]| \leqq |c/[3 - x_n^2]|, \tag{11}$$

that is,

$$|x_{n+2}| \leqq |x_{n+1}| \tag{12}$$

provided

$$|x_{n+1}| \leqq |x_n| < 3. \tag{13}$$

But (10) is a case in which (13) holds, which completes the induction. Obviously x_n is positive since

$$x_n = c/[3 - x_{n-1}^2] \quad \text{and} \quad |x_{n-1}| \leqq c/2 \leqq 1.$$

We have now proved that r_2 is a root of (3), r_2 being the limit of the function built up by means of the recursion formula

$$x_n = c/[3 - x_{n-1}^2], \qquad x_1 = 1. \tag{14}$$

In practice the approximation of this root can well be made by the following rule:

Subtract unity from three, divide the difference into c, square the quotient, subtract the result from three, divide the difference into c, etc.

[1] Geometrically this is equivalent to holding fixed the point whose abscissa is zero and moving the point whose abscissa is x_n along the curve toward the x-axis.

We will write r_2 in the following form:

$$r_2 = c/3 - [c/3 - [c/3 - \dots [c/3 - [c/3 - [1]^2]^2]^2 \dots]^2]^2. \tag{15}$$

Since a change in the sign of c merely changes the sign of r_2, we can disregard the restriction that the constant term of (1) must be positive and say that the above expression multiplied by $\sqrt{[-p/3]}$ is a root of (1) whatever be the sign of the constant term. Obviously the formula holds when $c = 0$.

Solutions[1] *for* r_1 *and* r_3. In order to secure r_1, add 2 to the roots of (4), obtaining

$$x^3 - 6x^2 + 9x - 2 + c = 0. \tag{16}$$

The root of this equation which is equal to $r_1 + 2$ we will denote by r_1'. Then $0 \leqq r_1' \leqq 1$. Applying formula (6), temporarily assuming that $c - 2 \neq 0$, we have

$$\begin{aligned} x_{n+1}/x_n = f(0)/[f(0) - f(x_n)] &= [-2 + c]/[-x_n^3 + 6x_n^2 - 9x] \\ &= [2 - c]/[x_n(x_n - 3)^2], \end{aligned} \tag{17}$$

whence

$$x_{n+1} = [2 - c]/(3 - x_n)^2,$$

and essentially the same procedure as we used in securing r_2 gives[2]

$$r_1' = (2 - c)/0^+ - [3 - [(2 - c)/0^+ \dots [3 - [(2 - c)/0^+ - [3 - 1]^2]^2]^2 \dots]^2]^2. \tag{18}$$

We will denote this function of c by $I(2 - c, 0, 3, 1)$. Then

$$r_1 = I(2 - c, 0, 3, 1) - 2. \tag{19}$$

Interpreted literally this formula means: *Subtract unity from three, square the difference and divide the result into two minus c, subtract the result from three, square and divide into two minus c, etc., and subtract two from the last approximation.*

In order to solve for r_3, subtract[3] 2 from the roots of equation (4), change the signs of the roots in the new equation, then proceed as when solving for r_1'. This gives

$$\begin{aligned} r_3' &= (2 + c)/0^+ - [3 - [(2 + c)/0^+ \dots [3 - [(2 + c)/0^+ - [3 - 1]^2]^2]^2 \dots]^2]^2, \\ &= I(2 + c, 0, 3, 1) = -r_3 + 2. \end{aligned}$$

Whence

$$r_3 = -I(2 + c, 0, 3, 1) + 2. \tag{20}$$

[1] We could of course divide out $x - r_2$, then solve the resulting quadratic for the other two roots, but this is more tedious than using the direct solutions, when the extreme roots alone are desired.

[2] This notation conforms with the general notation used later.

[3] This transformation, rather than reducing the roots by unity, is necessary for if we start the solution by the latter plan it is immediately seen that the interpolation theorems fails.

Observing that

$$I(2 - c, 0, 3, 1) = - I(c - 2, 0, 3, 1),$$

we can write the two roots, r_1 and r_3, in the combined form

$$\mp 2 - I(c \mp 2, 0, 3, 1). \tag{21}$$

In order to secure the extreme roots of (1) from this formula, we note that to change the sign of c merely interchanges the pair (21) and changes their signs, which is as it should be, since changing the sign of c merely changes the signs of the roots of (4). Thus, whatever be the sign of q, the extreme roots of (1) are

$$\pm 2\sqrt{-p/3} + \sqrt{-p/3}\, I(c \pm 2, 0, 3, 1), \tag{22}$$

where

$$c = q/\sqrt{-p/3}.$$

3. The Solution of Any Cubic Equation. Consider the equation

$$x^3 + px + q = 0. \tag{23}$$

If q be negative, change the sign of the roots. That is, we first solve the supplementary equation

$$x^3 + px + |q| = 0. \tag{24}$$

The first step is to locate one root between zero and some other limit. This is done by considering limiting values for p and q. Let r_1 be the root we are seeking, and x_1 be the other limit of this root. Since $f(0) = |q|$, x_1 must be such that $f(x_1) < 0$. Hence, if q be zero, we must have

$$x_1^3 + px_1 = x_1(x_1^2 + p) < 0.$$

Whence, if p is positive, x_1 must be negative, while if p is negative, x_1 must be equal to or less than $\sqrt{|p|}$. Hence $-\sqrt{|p|}$ is the largest value that x_1 can have for all values of p, and q equal to zero. Now let p be zero. Then x_1 must be equal to or less than $-\sqrt[3]{|q|}$. This suggests the limit $-\sqrt{|p|} - \sqrt[3]{|q|}$. And actual substitution in (24) gives

$$-(|p| + p)\sqrt{|p|} - 4|p|\sqrt[3]{|q|} - 3\sqrt{|p|}\sqrt[3]{|q|^2}$$

which is always negative. Hence [1]

$$-(\sqrt{|p|} + \sqrt[3]{|q|}) < r_1 \leqq 0.$$

[1] Not more than one root lies in this interval for (24) cannot have three negative roots. However, this fact is not used in the following solution.

See "An inequality for the roots of an algebraic equation" by J. L. Walsh, *Annals of Mathematics*, vol. 25, no. 3, p. 285, in which he determined the limits for the roots, in the complex plane.

We exclude the case $f(-\sqrt{|p|}-\sqrt{|q|})=0$, for in this case the hypotheses of our interpolation theorem do not hold. See (27). However, the formula that will be obtained will be seen to give the root in this case also.

Now add $\sqrt{|p|}+\sqrt[3]{|q|}$ to the roots of (24). The new equation is

$$(x-k)^3+px-pk+|q|=0, \tag{25}$$

where

$$k=\sqrt{|p|}+\sqrt[3]{|q|} \gtreqless 0. \tag{26}$$

We seek the root r_1' such that $r_1=r_1'-k$. Now write the linear interpolation formula

$$\begin{aligned} x_{n+1}/x_n &= f(0)/[f(0)-f(x_n)] \\ &= [k^3+pk-|q|]/[x_n^3-3x_n^2k+3x_nk^2+px_n]. \end{aligned} \tag{27}$$

Multiplying both members by x_n and completing the square in the denominator on the right gives

$$x_{n+1}=[k^3+pk-|c|]/[p+3k^2/4+(3k/2-x_n)^2]. \tag{28}$$

We will need to compare this identity with that obtained by substituting $n-1$ for n, namely

$$x_n=[k^3+pk-|q|]/[p+3k^2/4+(3k/2-x_{n-1})^2]. \tag{29}$$

In order to prove that the interpolation theorem holds, we need only show that, for all values of n, x_n is positive and never increases. For x_n then approaches a limit $\neq 0$ and hence x_n/x_{n-1} approaches unity. Obviously x_n is positive if $x_{n-1} \gtreqless k$, which is true if x_n never increases, for $x_1=k$. We will prove by induction that x_n never increases. From (28) and (29) we see that, if

$$|x_n| \gtreqless |x_{n-1}| \lesseqgtr 3k/2, \tag{30}$$

then

$$|x_{n+1}| \gtreqless |x_n|. \tag{31}$$

The induction will then be complete when we shall have verified (30) for a particular value of n. Since the upper limit of the root is k, we start with k $(=x_1)$. Putting $n=2$ and $x_1=k$ in (29) gives

$$\begin{aligned} x_2 &= [k^3+pk-|q|]/[p+3k^2/4+(3k/2-k)^2] \\ &= [k^3+pk-|q|]/[p+k^2]. \end{aligned} \tag{32}$$

Whence

$$|x_2|<|x_1|,$$

for the right member of (32) is easily seen to be less than k.

Using (29) as a recursion formula, we write

$$r_1{}^1=c_1/c_2{}^+[c_3-[c_1/c_2{}^+[c_3-\ \ldots\ [c_1/c_2{}^+[c_3-[c_1/c_2{}^+[c_3-k]^2]^2]^2]^2\ \ldots]^2]^2]^2, \tag{33}$$

where

$$c_1 = k^3 + pk - |q|, \tag{34}$$
$$c_2 = p + 3k^2/4, \tag{35}$$
$$c_3 = 3k/2, \tag{36}$$
$$k = \sqrt{|p|} + \sqrt[3]{|q|}. \tag{37}$$

We will denote the second member of (33) by

$$I(c_1, c_2, c_3, k). \tag{38}$$

Then one root of (24), namely r_1, is $I(c_1, c_2, c_3, k) - k$ Now a change of sign in the constant term of (23) merely changes the signs of the roots. Therefore, in all cases, one root of (23) is [1]

$$[q/|q|][I(c_1, c_2, c_3, k) - k]. \tag{39}$$

Factoring x minus this root out of (23), solving the resulting quadratic and simplifying the results, gives for the other two roots

$$-[q/2|q|][I(c_1, c_2, c_3, k) - k] \pm [1/2]\sqrt{-3[I(c_1, c_2, c_3, k) - k]^2 - 4p}. \tag{40}$$

4. Comments. If q be zero and p be negative, the hypotheses of the interpolation theorem do not then hold. However, actual substitution of $q = 0$ shows that (39) and (40) still hold if we take the precaution of dividing the common factor $p + k^2$ out of numerator and denominator in the first approximation, see (32), before applying the formula. This exception to the general solution can be avoided by taking for k a rational number larger than $\sqrt{|p|} + \sqrt[3]{|q|}$, preferably as nearly equal to the latter as can be easily discerned. Moreover, the extra steps required to attain the same degree of accuracy with the larger $|k|$ are not in general as tedious as the evaluation of $\sqrt{|p|} + \sqrt[3]{|q|}$. Inspection will show that such a choice of k would in no way invalidate the derivation of the above formulæ for the roots.

ILLUSTRATIVE EXAMPLE. Solve $x^3 - 3x + 1 = 0$. Using formula [2] (15),

$$c/(3 - [1]^2) = .5, \qquad c/[.5]^2 = 4/11 = .3636\ldots,$$
$$c/(3 - [4/11]^2) = .3487 +, \qquad c/(3 - [.3487 +]^2) = .3474 +,$$

and the next four approximations are

.34730625, .347297184, .347296424

and

.34729636.

The last approximation is correct to eight places. Substitution in the left member of (a) gives a remainder of .000 000 012 +. Using formula [3] (19), five approximations give for a second

[1] In the trivial case $q = 0$, it is necessary to define $q/|q|$ as zero.

[2] This is of the type $I(c_1, c_2, c_3, k)$, for it can be written $iI(ic, 3, 0, i)$ where $i = \sqrt{-1}$. The function I is of interest *per se*. The study of this function, when p and q have the continuum for ranges, will be taken up in a later paper.

[3] Instead of using (19), we could have gotten the same formula by putting $p = -3$, $q = 1$ and $k = 2$ in (39). This value of k is less than $\sqrt{3} + \sqrt[3]{1}$ and gives a more rapid approach to

root

$$-1.87938-.$$

Using (2), eleven [1] approximations give the third root,

$$1.531993+.$$

The sum of the three roots should be zero and it is .000 091 −.

POINTS ON THE CIRCUMCIRCLE.

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There are several methods by which definite points on the circle circumscribing a triangle may be associated with a given straight line in the plane of the triangle; some of these methods are discussed in the first three sections of this note. Few of the facts in this part of the paper are new, but the arrangement, the point of view and most of the analytical results are original. If one of the remarkable lines of the triangle be taken as the given line, an associated point may be worthy of notice among the remarkable points of the triangle. Some of these points are listed in the fourth section; among them are two notable points and another point which has been mentioned; but there is considerable new material in this section. Sections 5 and 6 are entirely new, so far as I am aware.

1. It is well known that every point on the circumscribed circle of a triangle possesses a pedal (or Wallace or Simson) line [2] on which are the feet of the perpendiculars from that point to the sides of the triangle. It is also well known that if the lines, instead of being drawn perpendicular to the sides, are drawn so as to make any given angle with the sides the feet are again collinear.[3] All the lines so drawn envelop a parabola which has the given point at its focus.[4] Indeed if p, q, r are the trilinear normal coördinates of the point, and if a, b, c are the lengths of the sides of the triangle, the equation of the parabola is

$$a^4x^2/p^2 + b^4y^2/q^2 + c^4z^2/r^2 - 2b^2c^2yz/qr - 2c^2a^2zx/rp - 2a^2b^2xy/pq = 0,$$

and its directrix is

$$a\cdot\cos A\cdot qrx + b\cdot\cos B\cdot rpy + c\cdot\cos C\cdot pqz = 0,$$

where A, B, C are the angles of the triangle. It is easily verified that this directrix

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the root than would the latter or the next larger integer, 3. In fact the evaluation of all the formulæ of this article can be expedited by giving increments to each approximation, these increments having been estimated roughly from the differences between preceding approximations.

[1] The best general procedure would be to use formula (19) or (20) according as c is positive or negative, then solve the reduced equation for the other roots.

[2] Leybourn's *Mathematical Repository*, old series, II (1798), p. 111. Mackay, *Proc. Edin. Math. Soc.*, IX (1890–1891), p. 83. This MONTHLY (*1916*, 61).

[3] Poncelet, *Propriétés Projectives*, vol. 2, p. 261, 2d edit. (1866).

[4] Steiner, *Gergonne's Ann.*, XIX (1828). (Mackay, *l.c.*)

is parallel to the pedal line of the point, whose equation is

$$(a\cdot\cos A - p\cdot\sin A)qrx + (b\cdot\cos B - q\cdot\sin B)rpy + (c\cdot\cos C - r\cdot\sin C)pqz = 0.$$[1]

Conversely, each line in the plane is associated with a single point on the circumcircle; it is a member of one of these families of lines.[2] Let ABC be the triangle, and let any line cross the sides at P, Q, R respectively. Let the equation of this line be $lx + my + nz = 0$. Then if lines are drawn from P, Q, R making the angle θ with BC, CA, AB, it can be shown [3] that these lines are concurrent for but one value of the angle, namely, that given by

$$\tan\theta = \frac{a^2mn(bn - cm) + b^2nl(cl - an) + c^2lm(am - bl)}{a^2mn(n\cdot\cos B + m\cdot\cos C) + b^2nl(l\cdot\cos C + n\cdot\cos A) + c^2lm(m\cdot\cos A + l\cdot\cos B) - lmn(a^2 + b^2 + c^2)}.$$

Using this angle, the point of concurrency is

$$\frac{a}{l(bn - cm)}, \quad \frac{b}{m(cl - an)}, \quad \frac{c}{n(am - bl)}.$$

It is easily verified that this point lies on the circumcircle, whose equation is

$$ayz + bzx + cxy = 0.$$

The point, J, at which these lines concur can be located as follows. From A draw a line parallel to PQR cutting the circle at U. Then UP crosses the circle again at J. This may be proved analytically, or as follows. Angles AUJ, QPJ are equal; and angles AUJ, ACJ are equal; hence angles QPJ, QCJ are equal, and the points Q, P, C, J are concyclic. Hence angles JPC, JQC are equal, so JP, JQ make equal angles with BC, CA respectively. It is easily proved that if BV, CW are drawn parallel to PQR, cutting the circle at V, W, then VQ, WR cross the circle again at the same point J, and hence that the angle JRA is also equal to the angle JPC.[4]

2. Again, starting with a given line, a point on the circumcircle may be found whose pedal line is parallel to the given line.[5] The coördinates of this point are

$$\frac{1}{l - m\cdot\cos C - n\cdot\cos B}, \quad \frac{1}{m - n\cdot\cos A - l\cdot\cos C}, \quad \frac{1}{n - l\cdot\cos B - m\cdot\cos A}.$$

[1] Casey, *Analytical Geometry*, 2d edit. (1893), gives a different form on p. 136.

[2] I have not found this fact recorded in places where the pedal line theorems are collected.

[3] The analytical work is tricky and tedious here and also in obtaining the other results.

[4] This is new, but compare an example at the foot of p. 164 in Hutton's *Principles of Projective Geometry* (1913).

[5] The fact is stated by Alison, *Proc. Edin. Math. Soc.*, III (1884–1885), § 17, cor. 3. The construction which immediately follows is found in several places, for example in Richardson and Ramsay's *Modern Pure Geometry* (1893), p. 54.

This point may be located in three ways.

(*i*) Draw AU parallel to the given line cutting the circle at U; then draw UO perpendicular to BC cutting the circle at O; and O is the point required. For, if ON is drawn perpendicular to AB and if N is joined to L, the intersection of UO with BC, since OLB, ONB are right angles, O, N, L, B are concyclic and the angles NBO, NLO are equal. But since O, A, U, B are concyclic, the angles AUO, ABO are equal. Hence angles NLO, AUO are equal, and therefore LN is parallel to UA which was drawn parallel to the given line.

(*ii*) Draw AX perpendicular to the given line. Then draw AO so that the angles OAB, CAX are equal. This line crosses the circle at O, which is the point required.[1] For, draw AU parallel to the given line. Join UB and UO. Now angles OAB, OUB are equal. Hence angles CAX, OUB are equal. Also angles UAC, UBC are equal. Therefore the sum of angles CAX, UAC equals the sum of angles OUB, UBC. But AX is perpendicular to AU. Hence OU is perpendicular to BC, and therefore the point O is the same as that reached by the preceding construction.

(*iii*) Take any two points, S and T, on the given line. Draw lines from S and T respectively parallel and perpendicular to the three sides of the triangle and let corresponding lines intersect at A', B', C'; S and T are evidently extremities of a diameter of the circle which circumscribes the triangle $A'B'C'$, and it is easily seen that this triangle is similar to the triangle ABC. If lines are drawn from A, B, C parallel respectively to $B'C'$, $C'A'$, $A'B'$, these lines will meet at a point, O, on the circumcircle which is the same point as that found by the preceding methods.[2] For, if AO is drawn parallel to $B'C'$, angles BAO, $SC'B'$ are equal. Also, if AX is perpendicular to ST, and if $B'Z$ is also drawn perpendicular to ST, the angles CAX, $SB'Z$ are equal. Now $B'ST$ is the complement of $SB'Z$; so is $B'A'T$. Again, angles $SC'B'$, $SA'B'$ are equal. But $SA'B'$ is the complement of $B'A'T$. Hence the angles BAO, CAX are equal, and the point O is the same as that reached by the preceding construction.

Thus, by any one of the three methods given, it is easy to find a second point, O, on the circumcircle associated with the given line.

3. Again, if any point, S, whose coördinates are f, g, h be taken on the given line, and if AS, BS, CS meet the circle at U', V', W', then it is not hard to show that $U'P$, $V'Q$, $W'R$ are concurrent at a point,[3] O', on the circle, whose coordinates are a/fl, b/gm, c/hn. Thus to each point on the given line corresponds a point on the circumcircle. In fact, the J-point discussed in the first section is the O'-point corresponding to S at infinity.

4. Now if we take one of the well-known lines[4] connected with a triangle, points can be found on the circumcircle which are associated with these re-

[1] Lachlan, *Modern Pure Geometry* (1893), p. 68, ex. 4.

[2] This is a generalization of the well-known construction to find Steiner's Point by drawing parallels from the vertices to the sides of Brocard's First Triangle.

[3] A pure geometry proof of this theorem will be found in this MONTHLY (*1919*, 59).

[4] The coördinates and equations of the most important points and lines connected with the triangle are collected in Vigarié's "Inventoire," published in *Comptes Rendus*, 1887.

markable lines in the ways described above. Some of the points so obtained may be found to have enough remarkable properties to warrant their inclusion among the notable points of the triangle. Among them are the following:

(*i*) Point whose coördinates are

$$\frac{a}{b^2-c^2}, \quad \frac{b}{c^2-a^2}, \quad \frac{c}{a^2-b^2}.$$

This is the *J*-point of Lemoine's line (the radical axis of the circumcircle and Brocard's circle); it is the *O*-point of Euler's line;[1] it is the *O'*-point of the orthocenter on Euler's line, of the symmedian point on Brocard's diameter, of the centroid on the line joining the symmedian point to the centroid, and of the inverse point of the incenter on the line joining that point to the incenter. This point is the intersection of the circumcircle with the ellipse (the isogonal transformation[2] of the orthic axis), which has the symmedian point for center and which also circumscribes the triangle; it is harmonically associated[3] with Brocard's diameter; and it is the focus of Kiepert's parabola.[4]

(*i'*) The point diametrically opposite to the point just considered. Its coördinates are

$$\frac{1}{2a^4-b^4-c^4+2b^2c^2-a^2b^2-a^2c^2},$$

and two similar expressions. It is the *O*-point of the orthic axis; and the intersection of the circumcircle with the isogonal transformation of Euler's line.

(*ii*) Steiner's Point, whose coördinates are

$$\frac{1}{a(b^2-c^2)}, \quad \frac{1}{b(c^2-a^2)}, \quad \frac{1}{c(a^2-b^2)}.$$

This is the *O*-point of Brocard's diameter;[5] it is the *O'*-point of the symmedian point on the line joining that point to the centroid, of the incenter on the line joining that point to its inverse, of the orthocenter on the line joining that point to its inverse, and of the pole of the line joining Brocard's points on Brocard's diameter; it is also known as the intersection of the circumcircle with Steiner's ellipse (the isogonal transformation of Lemoine's line), having the centroid as center and which also circumscribes the triangle; it also has other properties.[6]

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[1] Gallatly, *Modern Geometry of the Triangle* (n.d.), p. 26.

[2] If lines *SA*, *SB*, *SC* are drawn joining a point, *S*, to the vertices, and if other lines are drawn from the vertices so that angles *SAC*, *BAS'*; *SBA*, *CBS'*; *SCB*, *ACS'* are respectively equal, these lines meet at a point *S'*, which is isogonally conjugate to *S*; and if *S* moves along a given line, the locus of *S'* is a conic which is the isogonal transformation of the given line.

[3] *I.e.*, lines from the vertices through the given point cut the opposite sides in points which together with the points in which the given line cuts these sides form harmonic ranges.

[4] Casey, *l.c.*, p. 458.

[5] Gallatly, *l.c.*, p. 102.

[6] Casey, *l.c.*, p. 451.

(*ii'*) The point diametrically opposite to Steiner's Point is known as Tarry's Point. Its coördinates are

$$\frac{1}{a(b^4 + c^4 - a^2b^2 - a^2c^2)},$$

and two similar expressions. This is the O-point of Brocard's diameter and also of Lemoine's line; it is the point where the circumcircle intersects Kiepert's hyperbola (the isogonal transformation of Brocard's diameter).

(*iii*) The point whose coördinates are

$$\frac{1}{b-c}, \quad \frac{1}{c-a}, \quad \frac{1}{a-b}.$$

This is the O-point of the line joining the circumcenter and the incenter; it is the O'-point of the symmedian point on the line joining that point to the incenter, of the centroid on the line joining that point to the incenter; it is also harmonically associated with the line joining the symmedian point to the incenter, and it is the intersection of the circumcircle with the ellipse which is the isogonal transformation of the antiorthic axis.

(*iii'*) The point diametrically opposite to the point just considered. Its coördinates are

$$\frac{1}{1 - \cos B - \cos C}, \quad \frac{1}{1 - \cos C - \cos A}, \quad \frac{1}{1 - \cos A - \cos B}.$$

This is the O-point of the antiorthic axis of the triangle.

(*iv*) The point whose coördinates are

$$\frac{a}{b-c}, \quad \frac{b}{c-a}, \quad \frac{c}{a-b}.$$

This is the J-point of the antiorthic axis; it is the O'-point of the incenter on the line joining that point to the symmedian point, and of the centroid on the line joining that point to the incenter.

5. From a consideration of some of these special cases, the following general theorem was discovered:

THEOREM: *The O-point of any straight line is diametrically opposite to the point of intersection of the isogonal transformation of the given line with the circumcircle.*

The theorem is easily proved analytically.

6. Finally, the following theorems concerning a complete quadrilateral may be noted:

THEOREM: *The focal point*[1] *of a complete quadrilateral is the J-point of each side with respect to the triangle determined by the other three sides.*

THEOREM: *The intersection of the circle circumscribed to three sides of a complete quadrilateral with the circumcentric circle*[2] *of that quadrilateral is the O-point of*

[1] Clawson, *Annals of Math.*, XX (1919), p. 233.

[2] Clawson, *l.c.*, p. 236, § 5b.

the fourth side of the quadrilateral with respect to the triangle determined by the other three sides.

These theorems follow at once from theorems in the article referred to in the footnotes.

LOCI OF POINTS OF A MOVING BODY WHICH ARE EXCEPTIONAL POINTS OF THEIR TRAJECTORIES.

By E. L. REES, University of Kentucky.

In this paper we shall make a study of the instantaneous loci of points of a moving body which are exceptional points of their trajectories. The advantages in kinematics of vector methods which are used here have been noted by the writer in previous articles.[1]

1. Curve of Inflections. Let us find first the instantaneous locus of points of the body which are points of inflection of their trajectories. We shall see that this locus is in general a curve of the *third order* which we shall call the *curve of inflections*.

At a point of inflection the curvature is zero and the acceleration normal to the curve is zero. In other words the acceleration has the direction of the velocity at a point of inflection. Consequently such points are subject to the condition[2]

$$\dot{r} \times \ddot{r} = (\dot{p} + \dot{q}) \times (\ddot{p} + \ddot{q}) = 0, \tag{1}$$

in which p is the position vector of a point P fixed in the body, q the vectors from P to the points of the body satisfying the condition and r the position vectors of these latter points. It is understood of course that dots indicate time derivatives.

We shall consider first the general case in which $w \times \dot{w} \neq 0$, where w is the angular velocity vector. Taking P at the center of acceleration,[3] we have for our condition $(\dot{p} + \dot{q}) \times \ddot{q} = 0$. Multiplying this equation by $\dot{p}\cdot$, we get

$$[\dot{p}\dot{q}\ddot{q}] = [w\dot{w}q]\dot{p}\cdot q + w\cdot\dot{p}(w \times q)^2 = 0, \tag{2}$$

a homogeneous equation of the second degree in q. The curve of inflections therefore lies on a quadratic cone with the center of acceleration as vertex and with one element parallel to the central axis. The center of acceleration is of course a point of this curve, *i.e.*, the curve passes through the vertex of the cone.

If we divide the equation $(\dot{p} + \dot{q}) \times \ddot{q} = 0$ by q^2 and let $q = \infty$, we get $[w\dot{w}q_1]q_1 + (w \times q_1)^2 w = 0$, the subscript indicating unit vectors. Obviously

[1] See this MONTHLY (*1923*, 290–296); also (*1924*, 131–135). For a synthetic treatment of the subject of this paper see Schönflies, *La Géométrie du Mouvement*, Paris, 1893, pp. 139–144.

[2] This condition is satisfied also by points whose velocities are zero and by those whose accelerations are zero if there are any such points; and there may be certain points other than points of inflection for which the velocity and acceleration have the same direction such as points of undulation. All such points are of course contained in the loci discussed in this article.

[3] See an article by Ziwet and Field, this MONTHLY (*1916*, 376–378).

this equation is satisfied when and only when q has the direction of w. Hence we may say that the curve of inflections contains one and only one point at infinity, namely, the infinitely distant point on the central axis.

The condition for a point of inflection may also be written $\ddot{q} = u(\dot{p} + \dot{q})$, or

$$\dot{w} \times q + w \times (w \times q) = u(\dot{p} + w \times q). \tag{3}$$

Each value of the parameter u determines in general a definite q, which locates a unique point of inflection. By letting u vary we get the locus of these points of inflection, *i.e.*, the curve of inflections. Recalling that $w \times \dot{w} \neq 0$, we may write the vector equation of this curve in the form

$$q = w \cdot qR + \dot{w} \cdot qS + w \times \dot{w} \cdot qT,$$

where R, S, T and w, $\dot{w}$, $w \times \dot{w}$ are reciprocal sets of vectors, and the coefficients are functions of the parameter u. These coefficients may be determined by multiplying equation (3) in turn by $w\cdot$, $\dot{w}\cdot$ and $w \times \dot{w}\cdot$. We find that these coefficients are of the form

$$u(au^2 + bu + c), \quad u(du^2 + eu + f), \quad gu,$$

in which a, b, c, d, e, f, g are scalar functions of $\dot{p}$, w and $\dot{w}$. Thus the locus is a third order curve.

We now discuss the special cases for which $w \times \dot{w} = 0$.

1. If $w \neq 0$ we may write $\dot{w} = kw$, and by taking P on the central axis we may also write $\dot{p} = lw$. Substituting these values of $\dot{w}$ and $\dot{p}$ in the equation $(\dot{p} + \dot{q}) \times (\ddot{p} + \ddot{q}) = 0$, we find

$$lw \times \ddot{p} + klw \times (w \times q) - lw^2(w \times q) + (w \times q) \times \ddot{p} + (w \times q)^2 w = 0. \tag{4}$$

Let us first assume that $w \times \ddot{p} \neq 0$ and $\dot{p} \neq 0$. Multiplying in turn by $w\cdot$ and $\ddot{p}\cdot$, we get respectively $w \times \ddot{p} \cdot Q = w^2$ and $\ddot{p} \cdot Q = l^{-1}\ddot{p} \cdot w^{-1} - k$, where $Q = (w \times q)^{-1}$. These two equations together with the additional relation $w \cdot Q = 0$ enable us to solve for Q in terms of the vectors reciprocal to w, $\ddot{p}$ and $w \times \ddot{p}$. We thus find

$$Q \equiv (w \times q)^{-1} = (l^{-1}\ddot{p} \cdot w^{-1} - k)(w \times \ddot{p})^{-1} \times w + w^2(w \times \ddot{p})^{-1}.$$

Let K represent the second member of this equation. Then $w \times q = K^{-1}$, and since $w \cdot K = 0$ this equation represents a line. In this case then the curve of inflections is a straight line. If $w \cdot \ddot{p} = 0$, this line is the axis of acceleration.

If $\dot{p} = 0$, then $l = 0$ and equation (4) reduces to $(w \times q) \times \ddot{p} + (w \times q)^2 w = 0$, which is obviously satisfied by $w \times q = 0$, *i.e.*, by points of the central axis.[1] This is the only solution unless $w \cdot \ddot{p} = 0$, in which case the equation $(w \times q)^{-1} \times \ddot{p} + w = 0$ defines $w \times q$ as the vector chords of a circle in a

[1] These points are points of zero velocity.

plane perpendicular to w, and the equation therefore represents a right circular cylinder. This case is illustrated by uniplanar motion.

Finally, if $w \times \ddot{p} = 0$, then again $w \times q = 0$ satisfies equation (4) and we get the central axis. Moreover $w \times q$ cannot be different from zero, otherwise the equation $w \times \ddot{p} \cdot Q = w^2$ would lead to a contradiction since $w \times \ddot{p} = 0$ and $w \neq 0$.

2. If $w = 0$ and $\dot{w} \neq 0$, the resulting equation $\dot{p} \times (\ddot{p} + \dot{w} \times q) = 0$ gives $\dot{w} \times q = \dot{p}^{-1} \times (\dot{p} \times \ddot{p}) + \mu \dot{p}$ if $\dot{p} \neq 0$. This represents a line parallel to $\dot{w}$ if μ be determined uniquely so as to satisfy the condition

$$[\dot{w}\, \dot{p}^{-1}\, \dot{p} \times \ddot{p}] + \mu \dot{w} \dot{p} \cdot = 0.$$

If $\dot{w} \cdot \ddot{p} = 0$, the line is the axis of acceleration.

If $\dot{w} \cdot \dot{p} = 0$ and $\dot{w} \cdot \ddot{p} \neq 0$, there would be no locus since $\mu = \infty$.

If $\dot{w} \cdot \dot{p} = 0$ and $\dot{w} \cdot \ddot{p} = 0$, the locus would be a plane perpendicular to $\dot{p}$ which would contain P if $\ddot{p} = 0$.

If $\dot{p} = 0$, the velocities of all points are zero and the equation is satisfied by every point.

3. Finally if $w = \dot{w} = 0$, our equation $\dot{r} \times \ddot{r} = 0$ reduces to $\dot{p} \times \ddot{p} = 0$, a relation independent of q. Hence there will be no locus unless the velocity and acceleration of some one point P are in the same direction (or one of them vanishes) in which case all points satisfy the condition.

We may now state the following

THEOREM: *The locus of points of the body which are points of inflection of their trajectories is in general a curve of the third order. If, however, the angular velocity and acceleration have the same direction, or if one of them vanishes, the locus is in general a straight line.*

2. Locus of Points of Contact of Stationary Osculating Planes. Let us now find the locus of points of the body at which the osculating planes of their trajectories are instantaneously stationary.[1]

Since the torsion $T \left(= \dfrac{[\dot{r}\ddot{r}\dddot{r}]}{(\dot{r} \times \ddot{r})^2} \right)$ of the trajectory $r = p + q$ is zero for such points, it follows[2] that

$$[\dot{r}\ddot{r}\dddot{r}] = [\dot{p} + \dot{q}\ \ddot{p} + \ddot{q}\ \dddot{p} + \dddot{q}] = 0.$$

[1] At such points the osculating planes have contact of order higher than the second. The order of contact is determined by the vanishing of the triple products of the derivatives of r, which is shown by the following Taylor's expansion:

$$\delta = [\Delta r \dot{r} \ddot{r}] = \{[\dot{r}\ddot{r}\dddot{r}](\Delta t)^3/3! + [\dot{r}\ddot{r}r^{(iv)}](\Delta t)^4/4! + \cdots\} |\dot{r} \times \ddot{r}|^{-1},$$

where δ is the distance from the osculating plane to a point of the curve near the point of contact. It is assumed in this expansion that $\dot{r} \times \ddot{r} \neq 0$. If this product is zero, a different pair of derivatives would have to be used.

[2] Conversely, if $[\dot{r}\ddot{r}\dddot{r}] = 0$ and the derivatives of r are not zero, the torsion is zero except at points of inflection. At such points the torsion is not zero unless also $[\dot{r}\ddot{r}r^{(iv)}] = 0$. See next footnote.

But $\dot{q}$, $\ddot{q}$ and $\dddot{q}$ are expressible linearly in terms of q. Hence the equation is generally of the third degree in q and represents a third order surface. Whence the

THEOREM: *The locus of points of the body at which the osculating planes of their trajectories are instantaneously stationary is in general a surface of the third order.*

If one point of the body is fixed and P is taken at this point, our equation becomes $[\dot{q}\ddot{q}\dddot{q}] = 0$, a homogeneous equation of the third degree in q. Hence if one point of the body is fixed, the locus of points for which the osculating planes of their trajectories are instantaneously stationary is a cubic cone. This is the cone of lines whose polar axes are stationary.

It is seen at once that points satisfying the condition for points of inflection, namely, $(\dot{p} + \dot{q}) \times (\ddot{p} + \ddot{q}) = 0$, satisfy also the condition $[\dot{p} + \dot{q}\ \ddot{p} + \ddot{q}\ \dddot{p} + \dddot{q}] = 0$. It follows that the curve of inflections lies on the surface represented by this equation.[1]

Assuming $w \times \dot{w} \neq 0$, the curve of inflections is thus a part of the intersection of this third order surface and a quadratic cone (Art. 1).

We note the following special forms of the third order surface:

If $w \times \dot{w} = 0$, $w \times \ddot{w} = 0$ but $w \neq 0$, all of the third degree terms vanish and the equation is of the second degree in $w \times q$. Hence the locus is a cylinder with elements parallel to the central axis and with a conic for directrix. If all of the derivatives of p are perpendicular to the central axis, the equation becomes an identity and all points satisfy it. This last case is illustrated by uniplanar motion.

If $w = 0$, the equation is $[\dot{p}\ \ddot{p} + \dot{w} \times q\ \dddot{p} + \ddot{w} \times q] = 0$, which represents a quadric surface in general; but if also $\dot{w} \times \ddot{w} = 0$, this locus is in general a plane.

If $w = \dot{w} = \ddot{w} = 0$, all points or no points satisfy the condition according as some one point P does or does not satisfy it.

3. Envelopes of Loci of Exceptional Points. If not only $T = 0$ but also $\dot{T} = 0$, we have in addition to the condition $[\dot{r}\ddot{r}\dddot{r}] = 0$ the further condition $[\dot{r}\ddot{r}r^{(iv)}] = 0$, and the locus is the curve of intersection of two third order surfaces. But the curve of inflections is on both of these surfaces. Consequently, if $w \times \dot{w} \neq 0$, the locus of points for which $T = \dot{T} = 0$ is a curve of the sixth order. It will be observed that the intersection of these two surfaces is the characteristic of the surface of points of contact of stationary osculating planes, which therefore consists of the curve of inflections and the sixth order curve; in other words, the locus of these curves as t varies is the envelope of this surface.

In general the osculating planes of points of the surface have contact of the third order with the trajectories of these points, but at points of the sixth order curve they have in general fourth order contact.

[1] It is easy to show that the osculating plane at a point of inflection, which is usually determined by the vectors $\dot{r}$ and $\ddot{r}$, has contact of the third order, or higher. However, the osculating plane in general is not stationary at a point of inflection; for, in order that the osculating plane be stationary the condition $(\dot{r} \times \ddot{r}) \times (\dot{r} \times \ddot{r})_t = 0$, or $[\dot{r}\ddot{r}r^{(iv)}] = 0$ must be satisfied. If this condition is met, the osculating plane has contact of the fourth order, or higher, at a point of inflection.

If $T = \dot{T} = \ddot{T} = 0$, we have the additional condition

$$[\dddot{r}\ddot{r}r^{(\mathrm{IV})}] + [\dddot{r}\dot{r}r^{(\mathrm{V})}] = \frac{d}{dt}[\dddot{r}\ddot{r}r^{(\mathrm{IV})}] = 0.$$

Among the intersections of the surface represented by this equation and the other two surfaces are those points at which the osculating planes have contact of the fifth order. The locus of these points as t varies is a part of the edge of regression of the envelope of the third order surfaces of points of third order contact and is the envelope of the sixth order curves of the points of fourth order contact. The locus of the other points of intersection of these surfaces is the envelope of the curves of inflections and it is easy to show that this curve is a part of the locus of points at which the osculating plane has contact of the fourth order.

We sum up these results in the following statements:

The envelope of the third order surface locus of points of the body for which the trajectories have, with their osculating planes, contact of the third order consists of two surfaces; one is the locus of the curves of inflections and contains the points for which the tangents to their trajectories have second order contact, *i.e.*, points for which the tangents are stationary; the other is the locus of points for which the trajectories have, with their osculating planes, contact of the fourth order.

The edge of regression of this envelope consists of two curves; one is the envelope of the curves of inflections and contains points for which the trajectories have stationary tangents and have contact of the fourth order with their osculating planes; the other is the envelope of the sixth order curves (each of which is the instantaneous locus of points whose trajectories have fourth order contact with their osculating planes) and contains points for which the trajectories have with their osculating planes contact of the fifth order.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The Department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. A GENERALIZATION OF A PROPERTY OF CONFOCAL CONICS.

BY NORMAN MILLER, Queen's University.

A common method of describing an ellipse—by means of a loop of string around two pegs—depends on the property that the sum of the focal distances of any point on an ellipse is constant. If, instead of being stretched around two

pegs, the string be stretched around an ellipse (supposed, for physical purposes, an elliptic cylinder), the locus traced will be a confocal ellipse. This theorem is due to Bishop Graves,[1] whose proof made use of the familiar property of two confocal conics that a tangent and normal at a point of the outer one bisect the angles made by the two tangents from this point to the inner conic. His proof, however, based on the so-called method of infinitesimals is unsatisfactory inasmuch as no enquiry is made regarding the orders of the infinitesimals.[1]

The theorem may be generalized, as is indeed suggested by Salmon. The following theorem includes that concerning the ellipse as a special case.

THEOREM: *If from an external point P two tangents be drawn to a curve which has at every point a definite curvature and if the excess of the lengths of the two tangents over the arc intercepted between their points of contact is constant, the locus of the point P is a curve having the property that the tangent at any point on it makes equal angles with two tangents from that point to the original curve.*

If the given curve is closed and is everywhere concave toward its interior, then the locus can be traced mechanically by winding a loop of string around the curve and holding a pencil in the loop so as to keep it taut.

Let $y = f(x)$ denote the given curve. The hypothesis requires that $f'(x)$ and $f''(x)$ exist at every point of the curve except possibly at a finite number of points where $f'(x)$ is infinite. Let (X, Y) be a position of the tracing point P and (x_1, y_1) and (x_2, y_2) the points of contact of tangents from (X, Y) to $y = f(x)$. The axes of coördinates may evidently be chosen so that $x_1 < X < x_2$ and that neither $f'(x_1)$ nor $f'(x_2)$ is infinite. Then we have for parametric equations of the locus of (X, Y)

$$Y - f(x_1) = f'(x_1)(X - x_1), \tag{1}$$

$$Y - f(x_2) = f'(x_2)(X - x_2), \tag{2}$$

$$\sqrt{(X - x_1)^2 + (Y - f(x_1))^2} + \sqrt{(X - x_2)^2 + (Y - f(x_2))^2} - \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}\,dx = k, \quad \text{a constant.} \tag{3}$$

By virtue of the first two the third equation may be written,

$$(X - x_1)\sqrt{1 + (f'(x_1))^2} - (X - x_2)\sqrt{1 + (f'(x_2))^2} - \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}\,dx = k. \tag{4}$$

Since $X < x_2$, the second term requires a negative sign in order to render the term positive. From these equations, involving the parameters x_1 and x_2, we may find dY/dX, the slope of the tangent at (X, Y). By differentiating (1), (2) and (4) in succession, and writing for brevity

$$\sqrt{1 + (f'(x_1))^2} = \alpha, \qquad \sqrt{1 + (f'(x_2))^2} = \beta,$$

[1] See Salmon, *Conic Sections*, London, 1900, p. 377; or Sommerville, *Analytical Conics*, p. 199.

we get

$$dY/dX - f'(x_1) = f''(x_1)(X - x_1)dx_1/dX. \tag{5}$$

$$dY/dX - f'(x_2) = f''(x_2)(X - x_2)dx_2/dX. \tag{6}$$

$$\left(1 - \frac{dx_1}{dX}\right)\alpha + (X - x_1)\frac{f'(x_1)f''(x_1)}{\alpha}\frac{dx_1}{dX} - \left(1 - \frac{dx_2}{dX}\right)\beta - (X - x_2)\frac{f'(x_2)f''(x_2)}{\beta}\frac{dx_2}{dX} - \beta\frac{dx_2}{dX} + \alpha\frac{dx_1}{dX} = 0. \tag{7}$$

When dx_1/dX and dx_2/dX are eliminated from (7) by means of (5) and (6), we get

$$\alpha + \frac{f'(x_1)}{\alpha}\left(\frac{dY}{dX} - f'(x_1)\right) - \beta - \frac{f'(x_2)}{\beta}\left(\frac{dY}{dX} - f'(x_2)\right) = 0,$$

or

$$\beta\left\{\alpha^2 + f'(x_1)\frac{dY}{dX} - (f'(x_1))^2\right\} - \alpha\left\{\beta^2 + f'(x_2)\frac{dY}{dX} - (f'(x_2))^2\right\} = 0.$$

Hence

$$\beta\left\{1 + f'(x_1)\frac{dY}{dX}\right\} - \alpha\left\{1 + f'(x_2)\frac{dY}{dX}\right\} = 0.$$

It follows that

$$\frac{dY}{dX} = \frac{\alpha - \beta}{\beta f'(x_1) - \alpha f'(x_2)}. \tag{8}$$

Now form the quotient

$$\left\{\frac{dY}{dX} - f'(x_1)\right\} \Big/ \left\{1 + \frac{dY}{dX}\cdot f'(x_1)\right\}$$

$$= \left\{\frac{\alpha - \beta}{\beta f'(x_1) - \alpha f'(x_2)} - f'(x_1)\right\} \Big/ \left\{1 + \frac{\alpha - \beta}{\beta f'(x_1) - \alpha f'(x_2)}\cdot f'(x_1)\right\}$$

$$= \frac{\alpha - \beta - \beta(f'(x_1))^2 + \alpha f'(x_1)f'(x_2)}{\alpha(f'(x_1) - f'(x_2))}$$

$$= \frac{\alpha - \beta - \beta(\alpha^2 - 1) + \alpha f'(x_1)\cdot f'(x_2)}{\alpha(f'(x_1) - f'(x_2))} = \frac{1 - \alpha\beta + f'(x_1)f'(x_2)}{f'(x_1) - f(x_2)'}.$$

Similarly

$$\left\{\frac{dY}{dX} - f'(x_2)\right\} \Big/ \left\{1 + \frac{dY}{dX}f'(x_2)\right\} = \frac{\alpha - \beta - \beta f'(x_1)f'(x_2) + \alpha(f'(x_2))^2}{\beta(f'(x_1) - f'(x_2))}$$

$$= \frac{\alpha - \beta - \beta f'(x_1)f'(x_2) + \alpha(\beta^2 - 1)}{\beta(f'(x_1) - f'(x_2))} = \frac{\alpha\beta - 1 - f'(x_1)f'(x_2)}{f'(x_1) - f'(x_2)}.$$

Since dY/dX is the slope of the tangent to the curve traced by (X, Y) and $f'(x_1)$ and $f'(x_2)$ are the slopes of the tangents from (X, Y) to the original curve, these results establish the theorem.

II. A Problem in Speculation.

By J. P. Ballantine, Columbia University.

Let us consider a particular speculation. For every dollar that A puts up, if fortune favors him, he will receive q dollars, and if fortune is unkind to him, he will lose what he puts up. The probability of success is p.

The *expectation* is pq for each dollar risked, and must be greater than 1 in order to make the speculation attractive. If A were to try to make a maximum the sum of his cash on hand plus his expectation, by risking the proper number of dollars, he would find in the case of an attractive speculation that this sum can always be increased by increasing the stake.

The *moral expectation*, due to D. Bernoulli,[1] is

$$p \log (a - ax + aqx) + (1 - p) \log (a - ax),$$

where a and x are defined as below. It is possible to make the moral expectation a maximum by a proper choice of the stake. Moral expectation was defined as a working hypothesis, and with no especial regard to speculation. There is no *a priori* reason why the value of x which makes the moral expectation a maximum should be the best value for a speculator to use. In the present paper it is proved to be the best value to use in the long run.

In the mathematical treatment of the problem we cannot let enter such considerations as A's personal wants. In practice it may be, for instance, that what A has is enough for his wants, and a speculation would involve risking the possibility of owning something which he wants in return for the chance of having more money than he has wants for. We assume that A is a confirmed gambler, having in mind only to increase his fortune in the long run. This he does by entering into one speculation (for simplicity) at a time, and of the simple type described in the first paragraph. It further simplifies matters if all of the speculations are identical, *i.e.*, have the same values of p and q. This involves the assumption that the best thing to do in a single speculation is the best thing to do in one of a sequence of identical speculations. We assume that the best amount to put up in a given speculation (p, q) is proportional to a (the number of dollars A has) and otherwise depends only on the values of p and q. Another person B with twice as much money as A should put up twice as much on the same speculation.

If x denotes the proportional part of A's money which he puts up, so that ax is the amount he puts up, after one speculation his wealth is either

$$a - ax + aqx \qquad \text{or} \qquad a - ax$$

according as he wins or loses. After $h + k$ speculations, of which h are wins and k are losses, regardless of the order in which they occur, A's wealth is

$$a(1 - x + qx)^h(1 - x)^k.$$

[1] Jordan, "On Daniel Bernoulli's 'Moral Expectation' and on a New Conception of Expectation," Monthly (*1924*, 183–190).

The problem, then, is to choose x so that the above expression is a maximum. It is the same thing if we make

$$(a - ax + aqx)^{h/(h+k)}(a - ax)^{k/(h+k)}$$

a maximum; for multiplying an expression by a fixed power of a and extracting a positive root would have no effect on the value of x which produces a maximum. From the definition of probability, as h and k increase, the ratio $h/(h+k)$ approaches p. Hence if A is considering a large number of speculations, and since he has no way of knowing in advance the exact values of h and k separately—in fact, he knows p only by making a conservative guess—he may as well use the expression

$$(a - ax + aqx)^{p}(a - ax)^{1-p}.$$

This latter is rendered maximum in turn by making its logarithm

$$p \log (a - ax + aqx) + (1 - p) \log (a - ax) \qquad (1)$$

maximum.

Expression (1) is the moral expectation of A. In general, the moral expectation of a speculator who has undertaken a speculation with several outcomes is defined by D. Bernoulli as the sum of the products of the probability of each outcome by the logarithm of the total amount of money he will have (not receive) in case of that outcome. The theorem which we have proved in the case of a speculation of only two outcomes, and which is easily extended to the case of several outcomes, is: *The best amount of money to venture in any speculation is that amount which makes the "moral expectation" a maximum.*

III. The Notion "Matrix of a Determinant."

By A. A. Bennett, University of Texas.

There can be no question that so far as historical development is concerned, the notion of a matrix as a hypercomplex number subject to combination with other matrices of a suitable type by addition or multiplication is an outgrowth and by-product of investigations in the theory of determinants. In this historical sense the matrix is subsequent to the determinant.

The use of such expressions as the rows of a determinant, its columns, the upper left-hand corner, the main diagonal, and others of similar sort, bear evidence to the feeling that where a determinant is given its matrix may be recognized. These expressions are certainly not thought of as mere forms of diction, mere figures of speech handed down from the past and devoid of immediate logical justification. To be sure some writers of an older school ignore the distinction between matrix and determinant but the use of these terms cannot be wholly attributed to a continuation of such a confusion. For example, Bôcher in his classical *Introduction to Higher Algebra* remarks with his usual clarity and in

accordance with orthodox precedents (page 21), "Although, as we have pointed out, square matrices and determinants are wholly different things, every determinant determines a square matrix, the *matrix of the determinant*, and conversely every square matrix determines a determinant, the *determinant of the matrix*."

The question arises as to just what we mean when we say that a determinant has a matrix. It is certain that a determinant may be defined from a (square) matrix and so may be thought of as having this matrix as its origin provided that we know the matrix in question. But suppose, if possible, the determinant is encountered in some connection in which many distinct matrices are involved, and divorced from all identifiable association with any one particular matrix of the set. Is there in reality one and only one matrix which serves as "its matrix"?

The determinant in all ordinary uses is of course a number. If it be given as a numerical constant, there is little or no meaning to the term, element of the determinant. It is only as a function of certain variables, presumably of its elements, that the determinant could possibly define any matrix. If the matrix is assumed to be subject to some further special restriction, the problem is altered, as when the matrix is required to be symmetrical, skew-symmetrical, orthogonal, and so forth, and when the variables are not the elements, but the elements themselves are regarded as functions of one or more independent variables that figure throughout, as in the case of the so-called lambda-matrices, still further modifications of the discussion must be made.

Suppose then that the elements are given as independent variables. One still asks how are these elements to be identified? Do we have a natural number, n, and two subscripts, i and j, $i, j = 1, 2, \cdots, n$, and is each element, e_{ij}, recognized not only as an element but as the one which has these particular subscripts? If we do, there is no question that the determinant identifies the matrix, for then the determinant occupies the anomalous position of being all of a matrix and much besides. But such a convention is certainly contrary to a natural interpretation of the statement that certain familiar transformations alter the matrix but leave the determinant unaltered. Furthermore such a notion of a determinant is wholly inconsistent with the current definition of the determinant as a multilinear integral rational function of a certain prescribed sort, consisting of the algebraic sum of a number of signed products, since the character of a rational integral function in relation to its arguments is completely determined by internal properties and cannot depend upon assumed notation not consequent to the functional relationship, or upon extraneous order relations derived from some other representation. Nor is such a notion in accord with general mathematical tradition as illustrated in the case of polynomials. A polynomial identifies its roots and if it be agreed that the leading coefficient shall be unity, the roots in turn identify the polynomial. No one will ask that the separate roots be recognized individually as listed in the order in which they were selected in the formation of the polynomial considered. On the other hand, as one might insist, a polynomial identifies its coefficients and the set of coefficients identifies the

polynomial. Here, however, the several coefficients must be taken in an assigned order but they are further recognized not merely as coefficients but as coefficients of given powers of the independent variable. The question seems clearly to be one of the group of transformations leaving the expression invariant. While an arbitrary permutation of the roots of a polynomial leaves the expression invariant, and the roots are therefore identified only up to an arbitrary permutation among them, that is, they are recognized merely as an unordered set, yet on the other hand no non-identical permutation of the coefficients of the general polynomial of a given degree leaves this polynomial unaltered, and therefore its coefficients are individually recognizable.

If we are to apply such a criterion to the case under discussion, it would seem that we are compelled to say that the determinant identifies its matrix only as being one of a set of matrices all having the same determinant—which is almost verbal tautology, and if desired we could add—and convertible one into the other by a suitable group of transformations. It remains for one merely to settle upon the particular group of transformations to be used.

In any development of the theory of determinants one comes across unitary transformations of various sorts each of which is said to leave the determinant unaltered. The matrix after the unitary transformation does not usually have the same elements as it did previous to the transformation. Should the entire group of unitary transformations be considered, the problem is too general to have much interest. We have, merely, $D(x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n}, \cdots, x_{n1}, x_{n2}, \cdots, x_{nn})$ containing n^2 unknowns, given as equal, say, to $D(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{n1}, a_{n2}, \cdots, a_{nn})$, and the problem is to find the values of the n^2 unknowns in terms of the values of the known a's. The D is a polynomial in its arguments and it is possible for this case to choose $n^2 - 1$ of the unknowns restricted only by certain natural inequalities and then to determine the final unknown to satisfy the given condition. In this plausible interpretation of the problem, a determinant falls far short of defining "its matrix."

Let us, then, greatly restrict the inquiry, so as if possible to give some appreciable content to the notion, matrix of a determinant. Suppose that a set of n^2 variables is given, say, $x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n}, \cdots, x_{n1}, x_{n2}, \cdots, x_{nn}$, with explicit subscripts, and a matrix is formed of these in the manner suggested by the subscripts, and, finally, the determinant, D, of this matrix, is found. Suppose, in this determinant expressed as a polynomial, some one else should replace each x_{ij} by one of the variables selected from a set, $y_1, y_2, \cdots, y_{n^2}$, so that these latter constitute merely a definite but unknown permutation of the x's. The problem might be to show how far it is possible to identify the original variables. In any one term of the expanded determinant only elements from different rows and different columns appear. By inspecting in turn each of the terms, it would be readily possible to subject the y's to a double classification, which we might call into lines of the first type and lines of the second type. The names of the two types are assigned wholly arbitrarily, but otherwise the classification is determined by the expanded form of the determinant. The

lines of one of these types will be columns and the lines of the other type will be rows, although which will be which is inherently ambiguous. In the determinantal expansion, there are terms of two sorts, positive and negative. As a consequence, half of the terms, namely the $n!/2$ negative terms, cannot serve as principal term (product of the elements down the main diagonal). Any one of the $n!/2$ positive terms may however serve. More generally we may observe that any even permutation of rows, of columns, or of both, or a complete interchange of all rows for columns or vice-versa, leaves elements as elements and does not alter the value of the determinant, as indeed is known to every one interested in this theory.

One may say that in the restricted sense of the term a determinant classifies its elements doubly, and thus serves to identify its own determinant minors, their complements, cofactors, and so forth. Half of the possible $n!$ choices of a set of n elements, one from each set of each of the two types, are eligible for principal term and the other half are not. Any further methods for identification of the matrix are consequences of these facts only. For example, the determinant $uy - xv$ with elements, u, v, x, y, is of any one of the four forms,

$$\begin{vmatrix} u & v \\ x & y \end{vmatrix}, \quad \begin{vmatrix} u & x \\ v & y \end{vmatrix}, \quad \begin{vmatrix} y & v \\ x & u \end{vmatrix}, \quad \begin{vmatrix} y & x \\ v & u \end{vmatrix}.$$

Would it not therefore seem more desirable when speaking of specific rows or columns to call these as *of the matrix* and not as *of the determinant*, and never to use the term *matrix of a determinant* except in the obvious case in which a particular matrix is already known and exhibited and is the patent and unambiguous source from which the given determinant under discussion was obtained?

RECENT PUBLICATIONS.

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REVIEWS.

Relativity and Modern Physics.[1] By G. D. BIRKHOFF, with the coöperation of R. E. LANGER. Harvard University Press, 1923. xi + 283 pages. Price $4.00.

Relativity is a subject which is due to the physicist. It was developed to meet the needs that have arisen from the modern refinements of physical theory and experiment. Most of the writers on this subject have presented it as a branch of physics, requiring, indeed, a large amount of mathematical technique, but supposed to depend chiefly upon its physical aspects for its significance. It is possible, however, to take the mathematical part and make of it as logical a development as that of non-Euclidean geometry, and a study of relativity

[1] Reviewed by L. P. Eisenhart in *Science*, December 28, 1923, pages 539–541, and by O. Veblen in *Bulletin of the American Mathematical Society*, vol. 30, pp. 365–367.

as a branch of pure mathematics, dispensing with extraneous circumstances and assuming ideal conditions, would be of great interest, and would, indeed, add much to an understanding of its applications.

On the other hand, the physicist's point of view is what the mathematician should cultivate, and the physical books on relativity, written for advanced students, tend to bring the two classes together, just as in their earlier courses they are brought together by the type of mathematical text-book that has come into use in recent years.

Professor Birkhoff stands among the foremost of our mathematicians, and when his book on this subject was announced some mathematicians may have hoped for a purely mathematical treatise. This is not what he has written. As the title indicates, it is a book on "Relativity and Modern Physics." It would appear from the preface that it is intended equally for the mathematician and the physicist. The "postulational method," to which attention is particularly called, is a mathematical method, but the phraseology of the book seems to be that of the physicist. The constant use of physical terms, and the attempts to justify the assumptions made, as in accord with physical experiment, and to show the physical significance of the results,—these things would seem to appeal especially to the physicist. But it is possible to select some of the mathematics and study it in a purely mathematical way.

The book is a work of great originality, with new results and especially with new methods and points of view. Indeed, as far as we have observed, results which are not new are always obtained by new methods. Even to one who has devoted considerable time to the study of relativity the contents of these pages will come as new matter, leading to a study that will eventually give new insight into the entire subject. The book is not, however, easy to understand, even in the mathematical parts. The author hopes that it "will prove serviceable as a text for an undergraduate course," but it seems to be more difficult than he thinks it is. We have attempted to use it with a class of graduate students, and now after further study we should like to try it again, not, however, for its help in the explanation of difficulties, but for its suggestiveness. We have ourselves mastered only portions, if any of it, and this paper will not be a comprehensive review, but only an attempt to explain some of these portions.

A "Review of Classical Physics" takes up the first chapter and it is in the second that the new theories are introduced. In this chapter and for a considerable portion of the book the discussion is confined to a space of one dimension or line. A particle[1] A is supposed to have a system of time called its *local time*, measured by an "ideal clock" that belongs to it and has certain physical properties. By this clock it determines directly the time of any event at which it is present.[2] The local time of one particle cannot be compared with the local

[1] The individual on this line is sometimes called an observer and sometimes a particle according to the point of view. In this paper we shall use the term particle in all cases.

[2] In a mathematical treatment the notion of clocks is not necessary. A particle's local time, instead of being determined by the physical nature of such a mechanism, may be assumed to be perfectly arbitrary, subject only to the usual order relations of time. Certain particular ways

time of another except when they happen to meet, nor does a particle's local time tell it directly about the time of an event at which it is not actually present. By means of light signals, however, it can determine what it may call this time, assuming that a flash of light is always reflected at the moment midway between the time of sending and the time of receiving the flash, and this time can be compared with any local time at the event.

The same light signal serves also to determine what the particle may call the position of the event, assuming that the distance is proportional to the interval of time taken.

These determinations are made as follows: A sends a flash at time t_1 by its local time to a second particle B and receives the reflected flash at time t_2. A then considers that the time at B when the flash reaches B is $t = (t_2 + t_1)/2$ and that the position of B at that moment is denoted by the abscissa [1] $x = (t_2 - t_1)/2$. The reflection of the flash at B is an event. t is called the *apparent time* and the numerical value of x the *apparent distance* of the event.[2] It would be well always to put in the words "according to A," or "with respect to A," because the apparent time of an event with respect to one particle may be very different from its apparent time with respect to another particle.

Any event may be thought of as determined by its time and position, t and x, with respect to a given particle A, or by the quantities t_1 and t_2. Either pair of quantities may be changed to the other at any time, and many formulæ are simpler when expressed in terms of t_1 and t_2.

Events can be plotted on a chart, t and x being taken as coördinates or t_1 and t_2. In fact, with proper units the same graph will serve for both kinds of coördinates, the axes of the two systems bisecting each other's angles. A particle, passing through a series of events, is represented by the locus of the points that represent these events, which is called the particle's *world-line*. The *velocity* of the particle according to A, dx/dt, is represented by the slope of its world-line, and in the t_1t_2 chart this is always positive. When the variables belong to the system that is based on A's local time the chart is called A's chart.

A second particle B has a local time of its own and a system of variables, t_1', t_2', t' and x', based on its local time in the same way that t_1, t_2, t and x are based on A's local time. B's time for an event (the apparent time according to B) will not generally be the same as A's, but the two systems are connected by transformation equations which may be expressed as equations connecting t'

of determining the local time of different particles may then be selected for special study because they are interesting, as well as others that we think may have been imposed on the world by the nature of "space-time."

[1] A constant factor c, depending on the unit of length, is put equal to 1.

[2] Einstein in his first paper (1905) starts with these two equations, which he writes

$$t_B - t_A = t_{A'} - t_B, \qquad 2AB/(t_{A'} - t_A) = c,$$

c being obtained thus as the velocity of light. *The Principle of Relativity.* Memoirs by Lorentz, Einstein, Minkowski and Weyl, translated by W. Perrett and G. B. Jeffery, London, 1923, page 194. The idea of determining distances and setting up a system of coördinates by means of light signals, assuming c to be given, is developed very clearly in a paper by E. V. Huntington, "A new approach to the theory of relativity," *Festschrift, Heinrich Weber*, Leipzig, 1912, pages 146–169.

and x' with t and x (as, for example, the Lorentz equations), or t_1' and t_2' with t_1 and t_2. Here the advantage of the latter letters is strikingly shown, for only t_1 and t_1' occur in one equation and only t_2 and t_2' in the other equation. The equations are linear and those expressing t_1 as a function of t_1' and t_2 as a function of t_2' are reduced to symmetrical form by the use of a quantity λ, which is introduced casually as a "factor of proportionality" (page 31), but turns out to be of great importance and is called the *characteristic function*. A more concrete meaning for λ comes from its relation to B's local time pointed out at the bottom of page 31. If we make the variables refer to events of B and let v denote the velocity of B with respect to A ($v = dx/dt$), then we shall have $dt_1 dt_2 = (1 - v^2)dt^2$ (see bottom of page 40), and as $dt'^2 = \lambda^2 dt_1 dt_2$, $\lambda\sqrt{(1 - v^2)}$ will be equal to the ratio of the differential of B's local time to the differential of A's time at B, or what A would call the rate of B's local time. This fact is referred to again on page 39, but not otherwise mentioned in connection with the discussion of λ, which is always called the "characteristic function."

It will appear in a moment that λ is equal to 1 in the space-time of relativity, or at least of the restricted theory. With the variables referring to events of B the differential of B's time as it appears to A takes the form $\sqrt{(dt^2 - dx^2)}$, and as B is any particle this is a general formula for the differential of local time in the restricted theory when the space considered is one-dimensional. $dt^2 - dx^2$ is the "fundamental quadratic differential form" in this case, and so our author calls it ds^2 instead of dt'^2; and then he carries the idea into the general theory and throughout the book describes s as local time. This is certainly easier to understand than such terms as "interval" or "separation" or "linear element" (for ds), used with the idea of representing something analogous to distance in ordinary space.

The author emphasizes the graphical method of representing all these quantities. Thus the equations connecting the two systems of variables based on A's and B's local times are said to express the distortion that makes one of two charts fit the other. For ourselves we do not find the consideration of these charts very helpful. It is difficult to become familiar with the Minkowski method of combining time with space as another dimension of just the same kind as the dimensions of space, and the difficulties of this theory may as well be postponed till the difficulties of relativity are mastered. Moreover, there are too many charts. Thus, B being any particle, it is better to think of λ without thinking of B's chart, using A's chart alone. Every particle will then have a definite world-line, a definite velocity v with respect to A, represented by the slope of this world-line, and a definite time-rate represented by $\lambda\sqrt{(1 - v^2)}$.

It is possible to go farther and drop A's chart too, regarding all events and the world-lines of all particles as represented on the chart of a single arbitrary particle otherwise unmentioned. Indeed, this is what the author seems to do in his study of the function λ in chapter III. This chart is referred to at the bottom of page 33 and in line 7 on page 36. There are transformation equations connecting A with this arbitrary particle, and a characteristic function, and A

has with respect to it a velocity and a time-rate, but for the most part it is not necessary to think of these. We may then consider any number of particles A, and for each of them the velocity and time-rate of a particle B with respect to it, without thinking of any chart except that of the given arbitrary particle to which everything is referred.

He speaks of λ as the characteristic function of A when it is really a function [1] of B; that is, its value depends on the velocity and position of B with respect to A and the rate of B's local time. When, however, we consider different particles A and set up for each a system of reckoning time and position, there will be for each of them a function λ of the variable particle B. It is, therefore, convenient to speak of λ as A's characteristic function, thinking perhaps of its form rather than its value as what is determined by A.

In the third chapter he studies the function λ and bases upon it a classification of different kinds of *space-time*, space-time being a complete system of reckoning local times. Supposing that the external world possesses a definite kind of space-time, he would try to find what it is by making certain sets of assumptions and determining from the results which is most probably correct.

In the first place he assumes that space-time is *isometric:* given any two events O and O' and a particle A at O with characteristic function λ, there is at O' at least one particle A' whose characteristic function is also λ, the same as for A. It does not seem necessary to say that O and O' are the epochs of A and A' (events of local time zero), nor, indeed, to mention O at all.

He makes another assumption (stated on page 35) that might have been included in his definition of isometric space-time: Given A and its characteristic function λ, there is also at O' at least one particle whose characteristic function would be λ if the positive and negative directions along their line were reversed, this reversal causing an interchange of the subscripts 1 and 2 and a change in the signs [2] of x and v.

Two kinds of space-time are of special interest, *æolotropic* space-time and *isotropic* space-time.

In æolotropic space-time there is at the event O' only one particle A' whose characteristic function is a given λ and only one whose characteristic function would be λ if the direction of positive and negative were reversed.

If the space-time is æolotropic, let us take an event O and let M be any

[1] We may call attention to a similar form of expression in connection with coördinates. On A's chart the coördinates of a point P are spoken of as the coördinates of A, and the coördinates on B's chart as the coördinates of B. Thus on page 49 equations (15) are described as expressing the relation between the coördinates of A and the coördinates of B (see also page 156). At the top of page 61 he does, indeed, speak of the coördinates of P, but further along on the same page he refers again to the coördinates of the observer, here called O.

[2] For a given particle B the value of A's characteristic function λ is the same whichever direction is positive, but when we consider λ as a function of the velocity and position of B a change in the signs of these two quantities will usually make it a different function. In the cases considered it turns out that λ is a function of the velocity only of B and not of its position. The assumption means that a particle A' can be taken in place of A with a characteristic function whose value for B is the same function of minus B's velocity with respect to A' that the value of λ is of its velocity with respect to A.

particle at O. As the assumptions apply to M and the event O there will be a particle M' at O (and only one) that will have the same characteristic function when the directions of positive and negative for one are the opposite of their directions for the other. M and M' are called homologous functions, each homologous to the other.

The chart on which the world-lines of these two particles are supposed to be represented is taken with O for origin. On it these are homologous world-lines and on it the t_1 and t_2 axes are also said to be homologous world-lines. Therefore, if the two given world-lines are rotated about O, they will rotate in opposite directions, and in some unique position they will coincide and be a self-homologous world-line.

An interchange of the two directions will involve an interchange of t_1 and t_2, but the question might be asked what are the characteristic functions of t_1 and t_2, or how do t_1 and t_2 satisfy the definition of being homologous. The following explanation may enable us to obtain the self-homologous particle and its world-line without using the t_1 and t_2 axes: M and M' are homologous particles at O. A particle at O different from M or M' will have a different characteristic function, and its homologous particle will also be different from M or M'. We may speak of varying[1] M and M'. Particles at the same event are distinguished only by their velocities, velocities with respect to the system of reference, represented by the directions of their world-lines on the chart of this system. If the velocity of M varies until it becomes the same as the original velocity of M', then the velocity of M' will become the same as the original velocity of M. If M varies continuously and we assume that M' varies continuously when M varies continuously, there will be some point in this process where M and M' have the same velocity and are the same particle, the particle which is called a self-homologous particle, its characteristic function being the same whichever direction is taken as positive.

It is shown that all self-homologous particles (the self-homologous particles at different events) are relatively at rest, and then, because they form a class that is "uniquely singled out," they may be regarded as absolutely at rest and other particles as in absolute motion (page 47). Thus even in the æolotropic space-time of classical physics the only ground for saying that a particular kind of time and space are absolute is the uniqueness of the set of particles that have this time and space!

In isotropic space-time there is absolutely no distinction between any two particles. Each has the same characteristic function, and from the equations it appears that each has numerically the same velocity with respect to the other. By applying the equations to the two particles themselves we find that the characteristic function must be equal to 1 (page 48), and so the local time of each will appear to the other to have a rate which is equal to $\sqrt{(1 - v^2)}$, where v is their

[1] This "varying" does not take time. We may turn our attention first to one pair of particles and then to another, but the particles at O are all there at the same time, O itself belonging only to a single instant of time.

relative velocity. Isotropic space-time is the space-time of relativity, or at least of the restricted relativity.

One or two matters of detail may be mentioned in connection with this part of the subject.

There seems to be some ambiguity in the phrase "future for a particle" used at the bottom of page 25. Apparently it refers here to an event at which the particle *could* be present if it should move just fast enough to get to the place of the event at the right time. In this sense any one event represented by a point in the first quadrant of A's chart is in the "possible future" for any one particle at A's epoch. But there are at least two other ways of interpreting the expression. This subject is referred to again on page 30.

The paragraph at the bottom of page 28 could have been placed at the end of section 12 on page 26 and used in the proof that follows. But we do not need section 12 at all for a geometrical proof of the relation between the two charts. We need only assume that straight lines correspond to straight lines, parallel lines to parallel lines, and, in particular, horizontal lines to horizontal lines and vertical lines to vertical lines. For then equal horizontal segments will correspond to equal horizontal segments, and all segments of this kind on one chart must be in a given ratio to the corresponding segments on the other chart, any two on one chart being proportional to the corresponding two on the other chart. This is proved, first, when they are commensurable, and then by the usual methods when they are incommensurable. It follows that t_1 is a linear function of t_1', and in the same way we prove that t_2 is a linear function of t_2'.

The explanation of the theory of tensors is clearer than the treatment of other topics, and this part of the book is the easiest to understand (at least for the mathematician). It comprises most of chapters VI and VIII. The symbolism for summation is fully explained and the author makes his formulæ much easier to read by agreeing to use Greek letters for summation indices and Roman letters for other indices. He assumes at first that the transformations are linear, so that the coefficients of the tensor transformation equations are constants, and the ordinary derivatives of tensors are themselves tensors (see bottom of page 81). This does not help very much, but, on the other hand, the results can be generalized without difficulty (page 108), and much of his later development of tensor analysis is based on the use of variables whose transformations are linear. He also assumes for the most part that the number of variables is two, but points out that the discussion, almost without change, applies to any number of variables (for an exception see pages 108–111).

Chapter VIII [1] takes up the general theory of tensors. The theory is based

[1] It is interesting to compare this part of Birkhoff's work with the treatment of the same subject by Eddington in his *Mathematical Theory of Relativity* reviewed by Philip Franklin in this MONTHLY (*1924*, 444). The two books were published at about the same time and cover about the same ground so far as the topics mentioned in this review are concerned, but scarcely any subject is discussed in the same way or any theorem given the same proof. In general, Eddington's methods seem to be more in the spirit of the tensor calculus, while Birkhoff follows more closely the methods of the older mathematical analysis. References to Eddington in this review will be to the *Mathematical Theory of Relativity*.

on a *fundamental quadratic differential form* $ds^2 = g_{\alpha\beta}dx_\alpha dx_\beta$. The three-index symbols of the first and second kinds are denoted by $\Gamma_{i,jk}$ and Γ^i_{jk}, $\Gamma_{i,jk} = (\partial g_{ij}/\partial x_k + \partial g_{ik}/\partial x_j - \partial g_{jk}/\partial x_i)/2$, and in connection with the fundamental form there is a system of *geodesics* whose differential equations are

$$\frac{d^2x_i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0. \tag{71}$$

Two sets of variables are connected by equations of the form

$$x_i = c_{i\alpha}\bar{x}_\alpha + c_{i\beta\gamma}\bar{x}_\beta\bar{x}_\gamma + \cdots. \tag{64}$$

To begin with he lets $c_{ij} = \delta^i_j$ where δ^i_j is used [1] to denote 1 when $i = j$ and 0 when $i \neq j$. The equations are then

$$x_i = \bar{x}_i + c_{i\alpha\beta}\bar{x}_\alpha\bar{x}_\beta + \cdots, \tag{72}$$

and the two systems of coördinates may be said to be *associated at O*.

If also $c_{ijk} = -\frac{1}{2}\Gamma^i_{jk}|_0$ the system of coördinates $\bar{x}_i$ will be one for which the first derivatives of the coefficients of the fundamental form, and therefore all the three-index symbols of both kinds, vanish at O (page 117). Our author describes such a system as *geodesic at O*, and for a system geodesic at O he uses the letter τ in place of x and h_{ij} in place of g_{ij}.

Finally, there is introduced a special system of coördinates that he calls *Riemannian coördinates*. For this system he uses the letter ξ in place of x and γ_{ij} for g_{ij}, and he defines it by writing $\xi_i = \partial x_i/ds|_0 s$, where s is the parameter in the equations of the geodesic from O through the point x_i. The coördinates x_i and the coördinates $\bar{x}_i$ connected by equations (72) have the same Riemannian system, and the term "associated" may well be applied to any two such systems of coördinates. Two Riemannian systems at O are connected by a linear transformation. Thus if x_i and $\bar{x}_i$ are connected by (64) the associated Riemannian systems will be connected by the equations $\xi_i = c_{i\alpha}\bar{\xi}_\alpha$.

The "admissibility" of these coördinates is then shown by deriving the transformation of the ξ's, which is

$$x_i = \xi_i - \tfrac{1}{2}\Gamma^i_{\alpha\beta}|_0\xi_\alpha\xi_\beta - \tfrac{1}{6}(\partial\Gamma^i_{\alpha\beta}/\partial x_\gamma - 2\Gamma^i_{\alpha\mu}\Gamma^\mu_{\beta\gamma})_0\xi_\alpha\xi_\beta\xi_\gamma + \cdots. \tag{79}$$

This is obtained from the solution of (71) in power series in s, in which occur the expressions given above in the definition of ξ_i.

The derivatives of a tensor satisfy the tensor transformation equations when the variables are connected by a linear transformation. Taking a tensor expressed in terms of any variables x_i, he transforms it by passing from the x's to the associated Riemannian system at a point O, differentiates it, and then transforms the derivative by changing back to the original variables. In this way is obtained an expression which satisfies the tensor equations at O, and so

[1] This is Einstein's notation. Eddington writes g^i_j for the same thing, associating it with g_{ij} according to the rules of the tensor calculus.

anywhere, O being any point. This expression is the *covariant derivative* of the given tensor. Thus one use of the Riemannian system is to give him the covariant derivative.

Another use is to give him a certain "fundamental tensor." Since a Riemannian system is a particular kind of geodesic system, the first derivatives of the γ's are zero at O, but the second derivatives are not zero, and when the coördinates are transformed back to general coördinates these second derivatives become the elements of important tensors.

In particular, certain differences of these second derivatives become the elements of what is called by our author the *Riemannian tensor* and written R_{ijkl}, this expression being the transform of

$$\left(\frac{\partial^2\gamma_{il}}{\partial\xi_j\partial\xi_k}-\frac{\partial^2\gamma_{jl}}{\partial\xi_i\partial\xi_k}\right)_0.$$

The tensor $R^l_{ijk}=g^{al}R_{ijka}$, contracted with respect to i and l, is the tensor that Einstein puts equal to zero for the condition of a gravitation field in empty space. It is called R_{jk} and may be written [1] $g^{\alpha\beta}R_{\alpha jk\beta}$.

The difficulties in the transformation that gives us the formula for R_{ijkl} are partly removed by the introduction of an intermediate system of coördinates, an associated system geodesic at O. This system is defined by the equations

$$\tau_i=x_i+\tfrac{1}{2}\Gamma^i_{\alpha\beta}\big|_0x_\alpha x_\beta, \tag{94}$$

the Γ's being formed from the fundamental form for the x's. These coördinates have the same Riemannian system as the x's and are connected with it by the equations

$$\tau_i=\xi_i-\tfrac{1}{6}\frac{\partial\Gamma^i_{\alpha\beta}}{\partial\tau_\gamma}\big|_0\xi_\alpha\xi_\beta\xi_\gamma+\cdots,$$

the Γ's here formed from the τ's, and equal to zero at O. The equations defining the τ's show that their third derivatives with respect to the x's are zero, and the last equations show that the second derivatives of the τ's with respect to the ξ's are zero at O.

Among all the Riemannian systems at a point O there are some that are *normal;* that is, some for which $\gamma_{ij}|_0=\delta^i_j$, so that at O the fundamental form reduces to $d\xi_\alpha{}^2$ (page 124). But some of the statements that Birkhoff makes about normal coördinates seem to be true of any Riemannian system, and we have almost come to the conclusion that he sometimes uses the term normal when he means simply Riemannian. In particular, the equations

$$\Gamma^i_{\alpha\beta}x_\alpha x_\beta=0, \tag{78}$$

[1] R_{ijkl} is what Eddington would write B_{kjil}, i and j forming one pair and k and l the other pair in the pairing of the indices. This difference in notation, though a little confusing, comes naturally from the difference in the method of obtaining these quantities. For R^l_{ijk} Eddington writes B^l_{kji} following Einstein. Einstein does not mention the former tensor, but calls the latter the Riemann-Christoffel tensor. The contracted tensor R_{jk} is written by Einstein (and Eddington) G_{jk}. Eddington, pages 72 and 81, Einstein, *Principle of Relativity*, pages 141 and 142.

which are the conditions that given coördinates be Riemannian, are referred to on page 210 as the conditions that the coördinates be normal. In some of his proofs he uses normal coördinates when Riemannian coördinates, or even coördinates geodesic at O, would have all the advantages of the normal coördinates, these advantages coming usually from the fact that the first derivatives of the g's are zero at the point. This is true, in particular, of the proof on page 136. The discussion on page 126 and the proof on page 210 require that the coördinates be Riemannian, since they make use of equations (78), but they do not need to be normal. The formula

$$\left[\frac{\partial \Gamma^i_{jk}}{\partial \xi_l} + \frac{\partial \Gamma^i_{kl}}{\partial \xi_j} + \frac{\partial \Gamma^i_{lj}}{\partial \xi_k}\right]_0 = 0 \tag{85}$$

is perhaps the most important formula for Riemannian coördinates, since it expresses the condition imposed on the second derivatives of the g's. The corresponding formula, written with the three-index symbols of the first kind, is also true, and so on page 126 it is sufficient that the coördinates be Riemannian.

We may add that the discussion beginning on page 128 would have to be modified only slightly if carried through without the assumption that $h_{ij}|_0 = \delta^i_j$.

Several writers have made use of coördinates for which the first derivatives of the g's all vanish at a given point, and Eddington has introduced a system with a still closer approximation to Birkhoff's Riemannian system; namely, a system for which the second derivatives of the g's satisfy equations (85). These are introduced by him, however, after the theory of covariant differentiation and the Einstein equations have been developed. Eddington calls them *canonical coördinates*, and in the series which express the values of the original coördinates by means of these the terms of the first, second and third degrees have the same coefficients as in the series which express their values in terms of Riemannian coördinates (equations 79 given above).[1]

Professor Birkhoff has been able to give simple proofs for some important theorems.

At the beginning of chapter XIII he proves that the equations $R_{ijkl} = 0$ are the sufficient as well as necessary conditions that a system of coördinates with a given fundamental form can be transformed so that the g's shall be constants. The proof is indirect and does not show how to get the transformation that will do this as does the proof given by Eddington (pages 76–77), but it is simpler, being based on Euler's theorem concerning homogeneous polynomials.

At the end of the chapter on tensor analysis he proves for a Riemannian system at O that we have at this point the four equations

$$\frac{\partial R^\alpha_i}{\partial \xi_\alpha} - \tfrac{1}{2}\frac{\partial R^\alpha_\alpha}{\partial \xi_i} = 0. \tag{104}$$

[1] Eddington also (pages 77–79) uses a particular geodesic system as intermediate between the original coördinates and his canonical coördinates, but in writing the transformation resulting from the combination of the two transformations (equations 36.8) he has put into one group of terms three-index symbols based on the original coördinates and into another group of terms those based on the intermediate coördinates.

The corresponding equations will then be true for any coördinates at any point if the symbol "∂" is understood to denote covariant derivative. These are "the four identical relations of the Einstein theory," which have attracted considerable attention. Three or four proofs of these equations have been published. The most direct and elegant is by A. E. Harward, published in *The Philosophical Magazine*, volume 44, pages 380–382. It depends, however, on certain complicated formulæ which Birkhoff does not use. The other proofs, like Birkhoff's, are verifications and are made possible by the use of coördinates geodesic at O.

It is stated by Eddington (page 81) and others, as well as by Birkhoff, that because of these four identical relations only six of the ten Einstein equations are independent. It is a little puzzling to see just how these identities affect the independence of the Einstein equations. The identities are linear and homogeneous in the R's and their first derivatives, and if the R's were all zero they would be satisfied. But if all but four of the R's are zero and these equations serve then to make the remaining four zero, it must be because the remaining four, but not their derivatives, occur in them and occur in such a way that the equations can be solved with respect to them.

Professor Birkhoff discusses this question on pages 219–220, but his discussion is very concise. At first he seems to have in mind the case of normal coördinates, for which $R_i^j = R_j^i$ at O. He writes the identities with the variables ξ and supposes "that we satisfy some six of the equations $R_i^j = 0$, omitting, for instance, the four equations $R_i^4 = 0$."

But at this point the author changes from ξ to x, that is, as we understand it, from normal coördinates to general coördinates. He says, "By means of the identity just written we find four differential equations in R_i^4,

$$\frac{\partial R_i^4}{\partial x_i} + \varphi_i = 0,$$

where φ_i is linear in R_j^4 with $j = 1, 2, 3, 4$." In the general case the expressions in (104) are covariant derivatives and so there are terms in the R's themselves as well as in their derivatives. Also in the general case there are sixteen of the functions R_i^j that are distinct and not ten. But the sixteen are connected by six linear homogeneous equations corresponding to the equations $R_{ij} = R_{ji}$, and in some cases at least it may be supposed possible to put six of the functions R_i^j for which $i \neq j$ equal to zero, and express the other six in terms of the four R_i^4, and in this way to reduce the four identities to equations in these four R_i^4 alone, linear and homogeneous in them and their first derivatives.

Suppose that in the equations so reduced derivatives of all four of the R_i^4 occur and that we can select one particular derivative of each function and solve the equations for the four derivatives chosen. If, for example, these four derivatives are $\partial R_i^4/\partial x_i$, then the equations can be written as above, but φ_i will usually contain terms in the other derivatives of the R's as well as terms in the R's themselves.

This seems to be the meaning, but the wording in the book would indicate that the derivatives $\partial R_l^i/\partial x_i$ are necessarily the ones to be chosen, or rather that they are the only derivatives that occur.

The determination of the four R's from the equations

$$\frac{\partial R_l^i}{\partial x_i} + \varphi_i = 0$$

will involve four arbitrary functions, and if these functions are taken as equal to zero the R_l^i will be zero. This may be worked out in detail after the manner of certain existence proofs (see C. Jordan, *Cours d'Analyse*, 2d edition, volume 3, chapter 3). A function is determined by its value and the values of all of its derivatives at a single point, say at the point O. If R_l^i and all of its derivatives that do not involve differentiation with respect to x_i are made to vanish at O, this being done for each of the four R_l^i, it will follow by virtue of the above equations and the equations that can be obtained from them by successive differentiation that all of the remaining derivatives of R_l^i will vanish at O, and, therefore, that R_l^i must vanish throughout space-time.[1]

He expresses the result by saying that "six of the differential equations $R_l^i = 0$, joined with four appropriate boundary conditions, are equivalent to the entire set of ten equations."

Professor Birkhoff now passes to the functions R_{ij}. This he can do, for the functions of one set are all zero whenever those of the other set are all zero. He concludes that "the general solution of the equations $R_{ij} = 0$ will involve at least four arbitrary functions."[2] This does not refer to the arbitrary functions introduced in the integration of the four identities, they being taken equal to zero. The equations $R_{ij} = 0$ involve ten g's, and if there are only six of these equations four of the g's could be put equal to arbitrary functions and still leave as many unknowns as there are equations. These arbitrary functions seem to be the ones referred to. Whether with the "boundary conditions" it would still be possible to make these four functions entirely arbitrary might be a question. But the functions more or less arbitrary that are introduced in the integration of the six equations $R_{ij} = 0$ will perhaps take care of the "boundary conditions."[3]

[1] The author does not require that any derivatives of R_l^i shall vanish at O as seems to be necessary, but only that the R_l^i themselves do this.

[2] These are supposed to provide for an arbitrary transformation of coördinates.

[3] In the existence proofs as usually given the equations are solved for the highest derivatives with respect to one particular variable, and it would seem as if the present problem could be treated very simply in that way, especially as there is a considerable choice of equations and unknown functions. Moreover the functions $R_i{}^j$ may be used or the functions R_{ij}, and of the latter there are only ten.

To give only an outline of the method, let it be supposed, for example, that the six equations $R_{ij} = 0$, $i \neq j$, contain the six derivatives $\partial^2 g_{ij}/\partial x_1^2$, $i \neq j$, and can be solved for them, so that we have

$$\frac{\partial^2 g_{ij}}{\partial x_1^2} = \varphi_{ij},$$

With some breaks in the continuity it will be possible to include in this paper a mention of the three famous tests that have been proposed for the application of the Einstein theories to our universe, and to bring this study to a fitting climax.

Professor Birkhoff determines first (chapter XV) the form of a quadratic differential that will be unaltered by any space rotation about the origin, so that, so far as the quadratic is concerned, the space is symmetrical with respect to this point, supposed to be a unique point of attraction. Such a quadratic, expressed in polar coördinates, is shown to be of the form

$$\alpha dt^2 + 2\beta dt dr + \gamma dr^2 + \delta(d\theta^2 + \sin^2\theta d\varphi^2),$$

α, β, γ and δ functions of t and r at most. This form is then supposed to be reduced to

$$\lambda(dt^2 - dr^2) + \mu(d\theta^2 + \sin^2\theta d\varphi^2) \tag{153}$$

(page 245; see also page 109) and the coefficients λ and μ are made to satisfy the Einstein equations, $R_{ij} = 0$, for this form. At first it is assumed that λ and μ are functions of r alone and do not contain t, and under these conditions the Schwarzschild form is derived from the Einstein equations. Later (pages 253–256) it is shown that the above form can always be reduced by a transformation to the Schwarzschild form, so that, except for purposes of proof, it was not necessary to assume that λ and μ do not contain t.

This theorem is very important. It may be well to add to Birkhoff's proof one or two details.

He points out that f', g' and λ must not be zero at O, and says that this requirement is easily met because λ is not zero identically and there is only a single homogeneous relation between $f'(0)$ and $g'(0)$. Now he might have added

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where φ_{ij} contains the g's, their first derivatives, and their second derivatives except those for which the equations are solved.

With the six R_{ij} put equal to zero the four identities will contain only the four R_{ii} and their first derivatives. Suppose that they actually contain the four derivatives $\partial R_{ii}/\partial x_1$ and can be solved for them. Then if these four R_{ii} are made zero when $x_1 = 0$, whatever the values of the other x's, they will be zero identically and all of the Einstein equations will be satisfied.

Let the four functions g_{ii} be arbitrary functions of all of the x's. They can be regarded as given, or the equations as containing only the six unknown functions g_{ij}, $i \neq j$, and their derivatives. Put $\bar{g}_{ij} = g_{ij}|_{x_1=0}$ and $\bar{g}_{ij}' = \partial g_{ij}/\partial x_1|_{x_1=0}$. These are functions of x_2, x_3 and x_4, and when they are given the values of the six g_{ij} will be completely determined.

The four equations $R_{ii}|_{x_1=0} = 0$ involve the functions $\bar{g}_{ij}$ and $\bar{g}_{ij}'$, the first and second derivatives of the former and the first derivatives of the latter. They are differential equations for these twelve functions with x_2, x_3 and x_4 as the independent variables. All that is necessary, then, is to satisfy these equations, and this can be done without placing any restriction upon the g_{ii} as arbitrary functions of the four x's.

To carry through the solution thus outlined it is necessary first to determine the possibility of solving the equations in the manner indicated. Professor Birkhoff refers to this, pointing out that the equations must be compatible (see also Jordan, *l.c.*, page 303). If the solution can be completed we have a determination of the g's that involves four arbitrary functions of all the x's, functions necessary to make possible a most general transformation of coördinates, and then there will usually be arbitrary functions of three of the variables. It is to these that he seems to refer when he says, "the degree of indeterminateness would appear to be that of arbitrary functions of only three variables."

the reason, also necessary, that [1] μ_2 is not identically equal to μ_1 nor to $-\mu_1$; if it were we should have $\mu_{11} = \pm \mu_{12} = \mu_{22}$, and the equation $R_{33} = 0$ would be impossible.

In identifying the transformed quadratic with the Schwarzschild form he has three constants, k, $\bar{r}_0$ and m, that enable him to give preassigned initial values to $\bar{\lambda}$, $\bar{\lambda}_2$ and $\bar{\mu}_2$. But the initial value of $\bar{\mu}$ is also preassigned, and therefore there is a relation which these four values must satisfy if they are to be obtained by means of the three constants. This relation will be found to be

$$[\bar{\lambda}\bar{\mu}_2^2 + 2\bar{\mu}\bar{\lambda}_2\bar{\mu}_2 + 4\bar{\lambda}^2\bar{\mu}]_0 = 0.$$

With $\bar{\mu}_1 = 0$ at O it makes $\bar{\mu}_{11} = 0$ at O. Thus it might have been obtained, not as necessary to make a formal identification with the Schwarzschild form, but as necessary to make $\bar{\mu}_{11} = 0$ at O, and if at O $\bar{\lambda}$, $\bar{\mu}$, $\bar{\lambda}_2$ and $\bar{\mu}_2$ have assigned values satisfying this relation, then $\bar{\lambda}$ and $\bar{\mu}$ may be determined uniquely as functions of $\bar{r}$ alone, satisfying the equations $\bar{R}_{ij} = 0$, about as explained on page 255.

The relation given above may be expressed as a relation connecting the values of λ, μ, f' and g' at O, and when λ and μ are given it becomes an equation for $f'(0)$ and $g'(0)$ to satisfy in addition to the equation at the bottom of page 253.

In the solution of the two equations for $f'(0)$ and $g'(0)$ there will appear what seems to be a possible case of inconsistency; namely, when

$$\mu_1^2 - \mu_2^2 - 4\lambda\mu = 0.$$

λ being equal to $\frac{1}{2}(\mu_{11} - \mu_{22})$, this is a differential equation in μ whose solution may be expressed by saying that μ is equal to minus the square of the difference of two functions, one of $t - r$ and the other of $t + r$, while λ is equal to four times the product of their first derivatives. If these two functions are put equal to $\frac{1}{2}(\bar{t} - \bar{r})$ and $\frac{1}{2}(\bar{t} + \bar{r})$ the quadratic will reduce to the "isotropic" form (given at the bottom of page 245), a particular case of the Schwarzschild form. The reason that the transformation does not fail in this case is that the absolute term in the second of the two equations for $f'(0)$ and $g'(0)$ contains $\bar{\lambda}_2$ as a factor and this is zero in the present case, $\bar{\lambda}$ being equal to 1. Thus the two equations are not inconsistent, but are dependent, and the transformation desired is always possible.[2]

[1] In this paper subscripts will be given to λ and μ to denote derivatives with respect to t and r. The author uses the same letters, t, r, λ and μ, after the transformation that he uses before it, which is liable to cause some confusion. We will try to make our explanations clearer by writing $\bar{t}$, $\bar{r}$, $\bar{\lambda}$ and $\bar{\mu}$ for the new letters, and $\bar{R}_{ij} = 0$ for the new Einstein equations. Subscripts to $\bar{\lambda}$ and $\bar{\mu}$ will then denote derivatives with respect to $\bar{t}$ and $\bar{r}$.

[2] The above discussion will suggest that it may be possible to derive the Schwarzschild form directly by integrating the equations $R_{ij} = 0$ without assuming that λ and μ do not contain t, and thus dispense with the proof given on pages 253–256.

Take the equations

$$R_{11} + R_{22} = R_{33} = R_{13} = 0.$$

In the last chapter the author explains the three tests referred to, taking the plane $\varphi = \pi/2$ for the plane of operations.

The mathematical problem of the first test is to determine the motion of a particle moving so that $\int ds$ shall be an extremum. The resulting equations

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These are

$$\mu_{11} + \mu_{22} = \frac{1}{2\mu}(\mu_1^2 + \mu_2^2) + \frac{1}{\lambda}(\lambda_1\mu_1 + \lambda_2\mu_2),$$

$$\mu_{11} - \mu_{22} = 2\lambda,$$

$$\mu_{12} = \frac{1}{2\mu}\mu_1\mu_2 + \frac{1}{2\lambda}(\lambda_1\mu_2 + \lambda_2\mu_1).$$

Adding twice the third to the first and dividing by $\mu_1 + \mu_2$ we have

$$\frac{(\mu_1 + \mu_2)_1 + (\mu_1 + \mu_2)_2}{\mu_1 + \mu_2} = \frac{\mu_1 + \mu_2}{2\mu} + \frac{\lambda_1 + \lambda_2}{\lambda}.$$

Similarly, subtracting twice the third from the first and dividing by $\mu_1 - \mu_2$ we have

$$\frac{(\mu_1 - \mu_2)_1 - (\mu_1 - \mu_2)_2}{\mu_1 - \mu_2} = \frac{\mu_1 - \mu_2}{2\mu} + \frac{\lambda_1 - \lambda_2}{\lambda}.$$

These equations may be integrated as partial differential equations of the first order. For example, with u written for $\log(\mu_1 + \mu_2)/\lambda\sqrt{(-\mu)}$ the first reduces to $u_1 + u_2 = 0$, and its solution is of the form $u =$ function of $(t - r)$. The result of these integrations may therefore be expressed in the form

$$-(\mu_1 + \mu_2)f' = (\mu_1 - \mu_2)g' = \lambda\sqrt{(-\mu)},$$

f' and g' arbitrary functions of $t - r$ and $t + r$, respectively, which it is convenient to write as derivatives.

If λ and μ are given, f' and g' will be determined by these equations.

For the partial differential equation

$$\mu_1 \cdot (g' + f') - \mu_2 \cdot (g' - f') = 0,$$

one auxiliary equation is

$$\frac{dt}{g' + f'} + \frac{dr}{g' - f'} = 0;$$

or

$$g'(dt + dr) - f'(dt - dr) = 0;$$

whence

$$g - f = \text{const.},$$

and μ is a function of the single quantity $g - f$.

Therefore

$$\mu_1 = \mu' \cdot (g' - f') \quad \text{and} \quad \mu_2 = \mu' \cdot (g' + f'),$$

so that both equations above reduce to

$$\lambda\sqrt{(-\mu)} = -2\mu' f' g'.$$

The derivatives of μ_1 and μ_2 with respect to t and r, respectively, will be

$$\mu_{11} = \mu' \cdot (g'' - f'') + \mu''(g' - f')^2,$$

and

$$\mu_{22} = \mu' \cdot (g'' - f'') + \mu''(g' + f')^2;$$

whence

$$2\lambda = \mu_{11} - \mu_{22} = -4\mu'' f' g'.$$

Now the elimination of λ from the preceding equation leads to

$$\mu''\sqrt{(-\mu)} = \mu',$$

are the equations of the path in which it is supposed to move when free except for the attraction at the origin. Applied to the planet Mercury they give us the orbit and the motion of the planet in it, the result, as we know, in remarkable agreement with observation.

For the path of a ray of light ds is said to be equal to zero. The equations (157) and (158) were obtained as first integrals of the extremum problem. Multiplying the latter by Uf he points out that they are then true in the present case if the constant f is taken equal to zero. But how the deductions that follow (158) are valid now is not clear, the first step of these deductions being the elimination of the factors whose vanishing makes the equations true. Granted, however, that equation (159) is true with $f = 0$, the completion of the integrations is much like that of the preceding theory, and from the result is obtained the deflection determined by the Einstein theory.

The displacement of the spectral lines is discussed on the last two pages of the book. If we suppose that a vibrating atom of a specified element is the "clock" by which local time is determined, and if the differential of local time at any particle, as it appears to an observer whose system of coördinates is denoted by t, r, θ and φ, is represented by ds, then for the sun and earth

$$f_s : f_e = ds_s : ds_e,$$

where f_s denotes the number of vibrations received in a given interval of time from the atom on the sun and f_e the number received from the atom on the earth. But the frequency of vibrations received determines the position of the spectral lines, and therefore a difference in the rates of local time in the two places will cause a difference in the position of the spectral lines.

The Einstein theories have caused much discussion, some believing that the above tests have established the correctness of these theories, a few perhaps

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and on integration

$$\tfrac{1}{2}\mu' = -\sqrt{(-\mu)} + 2m,$$

$2m$ the constant of integration.

Put $\mu = -\bar{r}^2$ and therefore $\lambda = 4\bar{r}'f'g'$. The μ' equation becomes

$$\bar{r}' = 1 - 2m/\bar{r}.$$

$\bar{r}'$ means $d\bar{r}/d(g-f)$, so that $d(g-f) = d\bar{r}/\bar{r}'$; *i.e.*,

$$g'(dt + dr) - f'(dt - dr) = d\bar{r}/(1 - 2m/\bar{r}).$$

Put also

$$g'(dt + dr) + f'(dt - dr) = d\bar{t}.$$

The product of the sum and difference of these expressions will be

$$4f'g'(dt^2 - dr^2) = d\bar{t}^2 - [d\bar{r}/(1 - 2m/\bar{r})]^2.$$

But

$$\lambda = 4f'g'(1 - 2m/\bar{r}).$$

so that, finally, we have

$$\lambda(dt^2 - dr^2) = (1 - 2m/\bar{r})d\bar{t}^2 - d\bar{r}^2/(1 - 2m/\bar{r}).$$

This shows that (153) can always be reduced to the Schwarzschild form by the transformation

$$2g = \bar{t} + \int d\bar{r}/(1 - 2m/\bar{r}),$$
$$2f = \bar{t} - \int d\bar{r}/(1 - 2m/\bar{r}).$$

doubtful or actively opposed. This discussion does not touch the mathematician; we might almost say that to him it is meaningless. The mathematician finds that the mathematics is true. Some of it, the theory of tensors and tensor transformations, was in his hands long before the beginning of Einstein's work. These theories belong to analysis, but have sprung into prominence on account of their application to relativity. But there are other things in relativity, the Lorentz transformation, the FitzGerald contraction, the determination of the time and position of an event by means of light signals, and things considered in that branch of mathematics that we call kinematics, that may be treated mathematically without any reference to the external world, or to any physical causes producing them. This mathematics is of value to the mathematician apart from any physical application, just as are all of the various branches of mathematics in their relations to one another.

Personally we have found relativity a most baffling and elusive study. Time and again has it shown an open path to rapid progress, and time and again has the path suddenly closed with some stage of this progress not quite completed. Obstacles would remain perhaps for weeks or months; and then they would be removed and the way opened, only soon to be closed again by other obstacles.

In the explanations attempted here we have by no means covered all of the mathematics in Professor Birkhoff's book—there are still obstacles to a full completion of that task—and we have not touched the physics. But the explanations in this paper will serve, we hope, to show that the book is stimulating and suggestive. Whatever the cost, it is well worth study. It will contribute much to the advancement of mathematical and physical science, and a long time will pass before the store of ideas that it contains is exhausted.

H. P. MANNING.

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All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I. [*1924*, 145.]

The following is the program of the Mathematics Club of Brown University for the year 1924–1925:

November 14, 1924. "Linkages" by Professor Raymond Kurtz Morley.

December 12, 1924. "The Josephus problem" by Edson Clark Lockwood '25. "Caroline Herschel" by Lena Mae Dailey '26. "The four fours and related problems" by Leslie Thomas Fagan '26.

January 16, 1925. "Fibonacci series and related geometric paradox" by George Toyoharu Tsukuno '27. "Bhaskara's Lilavati" by Rose Mary Fogarty '25.

February 27, 1925. "A prismatoid formula" by William Elliot Cruise '26. "Abel" by Elizabeth Pearce Armstrong '27. "Lightning calculators" by Eugene Anderson Conant '27.

April 14, 1925. "Mathematics as an element in the history of thought." Open to all members of the University, by Professor Alfred North Whitehead, of Harvard University.

May 1, 1925. "Rational triangles and quadrilaterals" by John Joseph Bauer '25. "The Greek anthology" by Mary Virginia Kenny '26.

May (Date and placed to be announced later). Picnic.

(Report by Mr. G. Sauté.)

MATHEMATICS ROUND TABLE, Illinois Wesleyan University, Bloomington, Ill.

The following is a list of the subjects which were discussed in the Mathematics Round Table of the Illinois Wesleyan University:

"Curve tracing," "Fourth dimension," "Graphical machinery," "Magic squares," "The history of mathematics," "The history of Pi," "Linkages," "The life and work of Newton," "Negative and complex numbers," "Time and its measurement," "The theory of relativity," "Algebraic notations," "Mathematical prodigies," "Calculating machines," "René Descartes."

(Report by Mr. H. P. Bicknell.)

THE MATHEMATICS CLUB OF UNIVERSITY OF NORTH CAROLINA, Chapel Hill, N. C. [*1922*, 26.]

The officers of the club for the session 1923–1924 were:

Vinton A. Hoyle, president; C. H. Benson, vice-president; Professor E. T. Browne, secretary and treasurer.

The club was organized primarily for the graduate students in mathematics, for seniors, and the better students among the juniors. The attendance at all meetings was uniformly good.

The program was as follows:

October 23, 1923. "Nine proofs of the Pythagorean theorem" by V. A. Hoyle.
November 27, 1923. "Curiosities in numbers" by W. V. Parker.
January 29, 1924. "The history and development of logarithms" by C. H. Benson.
February 26, 1924. "Algebraic and geometric fallacies" by S. B. Smiethey.
April 7, 1924. "The parallel postulate" by G. S. Bruton.
May 27, 1924. "The trisection of the angle" by H. A. Breard. "Magic squares" by R. G. Florance.

(Report by Professor Browne.)

MATHEMATICS CLUB OF BUCKNELL UNIVERSITY, Lewisburg, Pennsylvania.
[*1922*, 177.]

The program for the meetings during the remainder of 1921–1922 were as follows:
March 27. "Algebraic fallacies" by Paul Schmidt '22. "Value of mathematics in high school" by Catherine Stahl '22.
May 8. "Division of the circle" by Professor G. L. Lowry.
May 13. Annual hike of the Club.
Election of officers: president, Kathryn Kinble '23; vice-president, Harold Schaefer '24; secretary-treasurer, Frieda Ebner '23.
1922–1923:
October 2, 1922. "Energy of the sun" by Professor William C. Bartol.
November 6. "New type of examination" by Frieda Ebner '23. "Practical value of high school geometry" by Harold Schaefer '24.
December 5. "Reasons why students fail in mathematics" by Mary E. Bailey '23. "Calculating prodigies" by Rachel Steckel '24.
January 8, 1923. "The status of mathematics in schools to-day" by Mrs. Anna C. Clark, Instructor in Mathematics.
February 5. "Relation of mathematics to commerce" by Chas. R. Birch '23. "Super-power survey" by Professor Rhodes '03, Professor of Electrical Engineering.
March 26. A mathematics bee with puzzles, questions, problems, symbols, definitions and magic properties and numbers, conducted by Professor H. S. Everett '12.
May 7. Reading: Leacock's "Education made attractive." Election of officers for the year 1923–1924: president, Harold Schaefer '24; vice-president, Hilda DeWitt '24; secretary-treasurer, Margaret Ackerman '25; faculty adviser, Professor H. S. Everett.
1923–1924:
November 19, 1923. Social meeting, mathematical games and puzzles.
January 14, 1924. "Teaching mathematics" by Professor Whyte, Professor of Oral English.
February 11. "Mathematics—A great inheritance" by Professor H. S. Everett.
May 19. "Roots of unity" by Professor J. S. Gold. Election of officers: president, Myron F. Decker; vice-president, Eugene D. Carstater; secretary-treasurer, Hulda J. Baxter.
May 24. Hike.

(Report by Professor Everett.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3125. Proposed by W. J. SIDIS, New York City.

If n is a prime number and r is a prime number not of the form $kn + 1$, then in the scale of radix r, a perfect nth power can be found ending in any given digit. Also, provided the given last digit is not 0, a perfect nth power can be found ending in any given set of digits.

3126. Proposed by V. M. SPUNAR, Chicago, Illinois.

Interpret the equations:

$$\begin{aligned} \sqrt{\alpha \sin A} + \sqrt{\beta \sin B} + \sqrt{\gamma \sin C} &= 0, \\ \sqrt{\alpha \cos A} + \sqrt{\beta \cos B} + \sqrt{\gamma \cos C} &= 0, \\ \sqrt{\alpha} \quad + \quad \sqrt{\beta} \quad + \quad \sqrt{\gamma} \quad &= 0. \end{aligned}$$

3127. Proposed by HARRY LANGMAN, New York City.

Assuming $x^3 + y^3 = z^3$ impossible, show that neither of the expressions $4q^3 - 27p^4$, $12q^3 - 3p^4$, can be squares.

3128. Proposed by J. ROSENBAUM, Milford, Connecticut.

Given the mid-points of the sides of a polygon, to construct the polygon.

3129. Proposed by C. N. SCHMALL, New York City.

Three hyperbolas are described each touching one side of a given triangle and having the remaining sides as asymptotes. Show that the product of their three latera recta is equal to the cube of the diameter of the inscribed circle of the triangle.

3130. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that if d, D are the double polar distances, or diameters, of the inscribed and circumscribed circles of a spherical triangle, and d', D' the corresponding diameters for its polar triangle,

$$d' + D = 180^\circ = d + D'.$$

3131. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment intercepted by two fixed tangents to a given conic on a variable tangent to the same conic.

3132. Proposed by J. W. CLAWSON, Ursinus College.

The trilinear coördinates of the circumcenter of a triangle are: $\cos A$, $\cos B$, $\cos C$; that is, the distances of this point from the sides of the triangle are proportional to $\cos A$, $\cos B$, and $\cos C$ respectively. The points ($\sin A$, $\sin B$, $\sin C$), ($\sec A$, $\sec B$, $\sec C$), and ($\csc A$, $\csc B$, $\csc C$) are also well-known points (symmedian, orthocenter and centroid) which can be located by simple geometrical constructions.

Can the points ($\tan A$, $\tan B$, $\tan C$) and ($\cot A$, $\cot B$, $\cot C$) be found by simple geometrical constructions? Have the points been named?

SOLUTIONS.

3081 [1924, 255]. Proposed by HARRY LANGMAN, New York City.

Show that

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 1 & \binom{4}{1} & \binom{6}{2} & \cdots & \binom{2m-2}{m-2} & \binom{2m}{m-1} \\ 0 & 1 & \binom{6}{1} & \cdots & \binom{2m-2}{m-3} & \binom{2m}{m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{2m}{1} \end{vmatrix} = 2,$$

where the parentheses represent binomial coefficients.

Solution by J. F. Reilly, University of Iowa.

From each element of the second row subtract half the corresponding element of the first row (the elements of the first row are even integers), and the given determinant becomes

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2k}{m-k} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 0 & 1 & \binom{5}{1} & \cdots & \binom{2m-2k-1}{m-k-2} & \cdots & \binom{2m-3}{m-3} & \binom{2m-1}{m-2} \\ 0 & 1 & \binom{6}{1} & \cdots & \binom{2m-2k}{m-k-2} & \cdots & \binom{2m-2}{m-3} & \binom{2m}{m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \binom{2m}{1} \end{vmatrix}.$$

In this determinant subtract each element of the second row from the corresponding element of the third row; in the resulting determinant subtract each element of the third row from the corresponding element of the fourth row; again in the resulting determinant subtract each element of the fourth row from the corresponding element of the fifth row, and so on, effecting the subtractions by using the theorem

$$\binom{n}{j} - \binom{n-1}{j} = \binom{n-1}{j-1}.$$

There will result finally the determinant

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2k}{m-k} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 0 & 1 & \binom{5}{1} & \cdots & \binom{2m-2k-1}{m-k-2} & \cdots & \binom{2m-3}{m-3} & \binom{2m-1}{m-2} \\ 0 & 0 & 1 & \cdots & \binom{2m-2k-1}{m-k-3} & \cdots & \binom{2m-3}{m-4} & \binom{2m-1}{m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \binom{2m-1}{1} \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

This determinant is equal to 2, since the cofactor of $\binom{2}{1}$, its upper left hand element, is unity.

Also solved by E. P. Bugdanoff, Theodore Bennett, and P. J. da Cunha.

3083 [1924, 305]. Proposed by Paul Capron, U. S. Naval Academy.

Show that if three distinct normals to the parabola, $y^2 = 4px$, are concurrent, the sum of their slopes is zero, and that if the sum of three numbers is zero, they are the slopes of concurrent normals to the parabola, $y^2 = 4px$. If two of the three concurrent normals are perpendicular and the third bisects the angle between them, show that they meet at $(3p, 0)$, $(8p, \sqrt{5}p)$ or $(8p, -\sqrt{5}p)$.

Solution by H. W. Bailey, Champaign, Illinois.

Let the normals be N_1, N_2, N_3, respectively, at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) on the parabola $y^2 = 4px$; their slopes are then $m_1 = -y_1/2p$, $m_2 = -y_2/2p$, $m_3 = -y_3/2p$. The condition that the sum of their slopes be zero is then $y_1 + y_2 + y_3 = 0$. Now the equations of the normals are $y_i x + 2py = (2p + x_i)y_i$, $i = 1, 2, 3$, respectively, for N_1, N_2, N_3. The necessary

and sufficient condition that these three normals be concurrent may be written after reduction as

$$\begin{vmatrix} y_1 & 1 & y_1^2 \\ y_2 & 1 & y_2^2 \\ y_3 & 1 & y_3^2 \end{vmatrix} = 0;$$

but if this be expanded and factored the result is $(y_1 - y_2)(y_2 - y_3)(y_1 - y_3)(y_1 + y_2 + y_3) = 0$. Hence the necessary and sufficient condition that three distinct normals be concurrent is that the sum of their slopes be zero.

Suppose that N_1 and N_2 are perpendicular and that N_3 bisects the angle between them. Then $m_2 = -1/m_1$, and $m_3 = (m_1 - 1)/(m_1 + 1)$ and imposing the condition that the sum of the slopes be zero we get the equation $m_1^3 + 2m_1^2 - 2m_1 - 1 = 0$. The roots of this equation are 1, $(-3 - \sqrt{5})/2$, $(-3 + \sqrt{5})/2$. Solving for m_2 and m_3, we get the sets of slopes $(1, -1, 0)$, $([-3 - \sqrt{5}]/2, [3 - \sqrt{5}]/2, \sqrt{5})$, $([-3 + \sqrt{5}]/2, [3 + \sqrt{5}]/2, -\sqrt{5})$. From these values and the original definitions of the slopes, we can compute the $x_1, y_1, x_2, y_2, x_3, y_3$ numerically; for example, for the first set of slopes they are, in order, $p, -2p, p, 2p, 0, 0$. The points of intersection are then found by solving two simultaneous equations for each set of roots, for example, the equations:

$$\begin{cases} x - y = 3p, \\ x + y = 3p, \end{cases} \quad \begin{cases} x(3 + \sqrt{5}) + 2y = (24 + 10\sqrt{5})p, \\ x\sqrt{5} \quad - \quad y = 7\sqrt{5}p, \end{cases} \quad \begin{cases} x(3 + \sqrt{5}) - 2y = (24 + 10\sqrt{5})p, \\ x\sqrt{5} \quad + \quad y = 7\sqrt{5}p. \end{cases}$$

Solving, we find the points of intersection to be $(3p, 0)$, $(8p, \sqrt{5}p)$, $(8p, -\sqrt{5}p)$ respectively.

NOTE BY THE EDITORS: A proof of the first statement follows readily from the fact that the cubic in m,

$$pm^3 + m(2p - x') + y' = 0,$$

lacks the m^2 term.

The condition that two of the normals be perpendicular and the third bisect one of the angles between them would be satisfied also if $m_2 = (1 + m_1)/(1 - m_1)$; but, since this can be reduced to the condition already considered by interchanging m_1 and m_2, it need not be discussed further.

Also solved by W. L. AYERS, THEODORE BENNETT, E. P. BUGDANOFF, MAURICE BAUDIN, R. W. BRINK, E. H. CLARKE, P. J. DA CUNHA, S. E. FIELD, C. N. MILLS, A. PELLETIER, and J. B. REYNOLDS.

3084 [1924, 305]. Proposed by H. S. UHLER, Yale University.

Given that a and b are constants, evaluate the integral

$$\int \{[x^3 \arcsin (b/x)]/\sqrt{a^2 - x^2}\}dx.$$

I. SOLUTION BY FREDRICK WOOD, Lake Forest College.

By integration by parts this integral becomes

$$\left[\frac{(a^2 - x^2)^{3/2}}{3} - a^2\sqrt{a^2 - x^2}\right]\arcsin\frac{b}{x} + b\int\left[\frac{(a^2 - x^2)^{3/2}}{3} - a^2\sqrt{a^2 - x^2}\right]\frac{dx}{x^2\sqrt{1 - \frac{b^2}{x^2}}}.$$

Then, by using the substitution

$$a^2 - x^2 = (a^2 - b^2)\sin^2\theta,$$

the second term can be put in the form

$$\frac{b(a^2 - b^2)}{3}\int\left[\sin^2\theta - \frac{2a^2}{a^2 - b^2} + \frac{2a^4}{(a^2 - b^2)(a^2\cos^2\theta + b^2\sin^2\theta}\right]d\theta.$$

After integrating, and changing variables, the final result is

$$\frac{1}{8}\left\{-2(x^2 + 2a^2)\sqrt{a^2 - x^2}\arcsin\frac{b}{x} - b(3a^2 + b^2)\arcsin\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - b^2}}\right.$$
$$\left. - b\sqrt{(a^2 - x^2)(x^2 - b^2)} + 4a^3\arctan\frac{b\sqrt{a^2 - x^2}}{a\sqrt{x^2 - b^2}}\right\} + C.$$

II. Solution by P. J. da Cunha, University of Lisbon.

An integration by parts applied to the proposed integral gives

$$I = -x^2\sqrt{a^2-x^2}\arcsin\frac{b}{x} + 2\int x\sqrt{a^2-x^2}\arcsin\frac{b}{x}\,dx - b\int\frac{x\sqrt{a^2-x^2}}{\sqrt{x^2-b^2}}\,dx.$$

Again applying the same method to the first integral of this result, we find, after combining several terms,

$$I = -\frac{(2a^2+x^2)\sqrt{a^2-x^2}}{3}\arcsin\frac{b}{x} - \frac{b}{3}\int\frac{(2a^2+x^2)\sqrt{a^2-x^2}}{x\sqrt{x^2-b^2}}\,dx.$$

Setting $t = \sqrt{a^2-x^2}/\sqrt{x^2-b^2}$, the integral on the left, omitting the factor $-b/3$, becomes

$$-(a^2-b^2)\int\frac{t^2[3a^2+(2a^2+b^2)t^2]}{(1+t^2)^2(a^2+b^2t^2)}\,dt.$$

The integrand is then decomposed into

$$-\frac{1}{(1+t^2)^2} - \frac{a^2+b^2}{(a^2-b^2)(1+t^2)} + \frac{2a^4}{(a^2-b^2)(a^2+b^2t^2)}.$$

The integration of these three fractions gives the final result:

$$I = -\frac{(2a^2+x^2)\sqrt{a^2-x^2}}{3}\arcsin\frac{b}{x} - \frac{b\sqrt{(a^2-x^2)(x^2-b^2)}}{6} - \frac{b(3a^2+b^2)}{6}\arctan\sqrt{\frac{a^2-x^2}{x^2-b^2}} + \frac{2a^2}{3}\arctan\frac{b}{a}\sqrt{\frac{a^2-x^2}{x^2-b^2}} + \text{constant}.$$

Also solved by H. W. Bailey, Theodore Bennett, C. K. Robbins and the Proposer.

3088 [1924, 306]. Proposed by C. J. Coe, University of Michigan.

If a sphere of radius $2r$ is cut by a circular cylinder of radius r so that the center of the sphere lies on the surface of the cylinder, show that the length of the curve of intersection of the two surfaces is twice that of the curve of intersection of this cylinder with a plane cutting its elements at an angle of 45°.

Solution by Maurice Baudin, Miami University.

With reference to rectangular axes, the origin being the center of the sphere, an element of the cylinder being the Z-axis, and u the angle which the radius r of the cylinder makes with the positive X-axis, we have as the equations of the curve of intersection

$$x = r(1+\cos u), \qquad y = r\sin u, \qquad \text{and} \qquad z = \pm 2r\sin(u/2).$$

If s_1 be the length of this curve, we have

$$\tfrac{1}{4}s_1 = \sqrt{2}r\int_0^{\pi}\sqrt{1-\tfrac{1}{2}\sin^2\frac{u}{2}}\,du,$$

or, setting $u/2 = \varphi$,

$$\tfrac{1}{4}s_1 = \sqrt{2}r\int_0^{\pi/2}\sqrt{1-\tfrac{1}{2}\sin^2\varphi}\,d\varphi.$$

The equation of the given ellipse may be written

$$x = r(1+\cos\varphi), \qquad y = z = r\sin\varphi,$$

and its length s_2 is given by

$$\tfrac{1}{4}s_2 = \sqrt{2}r\int_0^{\pi/2}\sqrt{1-\tfrac{1}{2}\sin^2\varphi}\,d\varphi.$$

Hence, $s_1 = 2s_2$.

Also solved by J. A. Bullard, Theodore Bennett, and J. B. Reynolds.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Mr. W. B. LORING, of the University of Maine, has been appointed assistant professor of mathematics at Colby College, to fill the vacancy caused by the illness of Professor B. E. Carter.

Professor E. A. SHAW, for twenty-five years professor of mathematics at Norwich University, has retired from active teaching.

Professor R. G. D. RICHARDSON, of Brown University, has been granted leave of absence for the second semester of the academic year 1924–1925.

Professor L. P. EISENHART, of Princeton University, has been elected dean of the faculty.

Dr. R. E. GLEASON, of Princeton University, has been appointed professor of mathematics at Temple University, Philadelphia.

Mr. D. H. MACPHERSON, of Brooklyn Polytechnic Institute, died December 28, 1924, at the age of twenty-two.

Professor A. G. HALL, registrar of the University of Michigan, died January 11, 1925.

Dr. W. E. STANTON, formerly professor of mathematics at Iowa State College, died recently at the age of seventy-four.

The first Carus Monograph has been ordered by sixty per cent. of all members who have thus far paid their dues. Each member is entitled to order one copy at the cost price either now or at any future time. All orders should be sent to Secretary W. D. Cairns, Oberlin, Ohio. Non-members should send their orders to the Open Court Publishing Company, Chicago, Illinois.

The following reports of Summer Sessions to be held in 1925 have been received:

University of Chicago, first term, June 20 to July 29; second term, July 30 to September 4. In addition to the usual courses in Trigonometry, College algebra, Plane and solid analytical geometry and Calculus given by regular members of the staff and by Professors H. E. BUCHANAN and H. L. SMITH, the following advanced courses are announced: By Professor G. A. BLISS: Calculus of variations; Thesis work in analysis. By Professor H. E. SLAUGHT: Differential equations; Definite integrals. By Professor E. T. BELL: Algebraic numbers; Elliptic and theta functions. By Professor F. R. MOULTON: Periodic orbits; Analytic mechanics. By Professor G. C. EVANS: Mathematical theory of economics; Integral equations and Potential theory. By Professor A. C. LUNN: Relativity and the quantum phenomena; Theory of functions of a complex variable. By Professor E. P. LANE: Analytical projective geometry; Metric differential geometry.

Columbia University, July 6 to August 14. In addition to courses in Trigonometry, Solid geometry, College algebra, Analytic geometry, and Calculus, and

a series of courses for teachers of secondary mathematics, the following advanced courses are offered: By Professor H. F. BLICHFELD: Elementary exposition of selected topics in modern mathematics; Theory of continuous groups. By Professor W. B. FITE: Introduction to higher algebra. By Professor G. A. PFEIFFER: Theory of functions of a real variable. By Dr. K. W. LAMSON: Differential equations.

Cornell University, July 6 to August 14. By Professor W. A. HURWITZ: Analysis. By Professor VIRGIL SNYDER: Projective geometry. The following reading and research courses are also offered: By Professor C. F. CRAIG: Functions of a complex variable. By Professor SNYDER: Algebraic geometry. By Professor F. R. SHARPE: Hydrodynamics and elasticity. By Professors D. C. GILLESPIE and HURWITZ: Analysis. By Professors W. B. CARVER and F. W. OWENS: Projective geometry.

University of Illinois, June 22 to August 15. In addition to the usual courses in College algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered: By Professor J. B. SHAW: Philosophy of mathematics. By Professor R. D. CARMICHAEL: Partial differential equations. By Professor E. B. LYTLE: Teachers course; History of mathematics. By Professor Arnold EMCH: Projective geometry. By Mr. L. L. STEIMLEY: Advanced calculus. By Mr. B. R. REINSCH: Theory of equations.

University of Iowa, first term, June 15 to July 24. In addition to courses in Algebra, Trigonometry, Analytic geometry, and Calculus, the following courses are offered: By Dr. D. H. MENZEL: Astronomy. By Dr. ROSCOE WOODS: Projective geometry. By Professor W. H. WILSON: Differential equations; Subject matter and teaching of mathematics. By Professor J. F. REILLY: Advanced algebra; Interpolation and graduation. Second term, July 27 to August 28. By Professor W. H. WILSON: Theory of equations. By Professor E. W. CHITTENDEN: Differential equations; Advanced calculus.

Johns Hopkins University, June 30 to August 7. In addition to courses in College algebra, College geometry, and Introductory calculus, the following graduate course is offered: By Professor F. D. MURNAGHAN: Functions of a complex variable with applications to differential equation theory.

University of Michigan, June 22 to August 14. In addition to courses in Algebra, Plane and solid geometry, Trigonometry, Analytic geometry, Calculus, Mathematical statistics, and the Mathematical theory of interest and insurance, the following advanced courses are offered: By Professor W. B. FORD: Advanced calculus; Determinants and the theory of equations. By Professor L. C. KARPINSKI: Teaching of geometry; History of mathematics. By Professor J. W. BRADSHAW: The figures of solid geometry. By Professor T. H. HILDEBRANDT: Theory of functions of a complex variable. By Professor T. R. RUNNING: Empirical formulas. By Professor PETER FIELD: Mechanics. By Professor H. C. CARVER: Theory of probability; Finite differences. By Professor NORMAN ANNING: Solid analytic geometry. By Mr. S. E. FIELD: Differential equations.

University of Minnesota, first term, June 22 to August 1; second term, August 2 to September 5. The department of mathematics will offer an intensive course entitled: Selected topics in advanced mathematics. The topics for 1925 are: First term: By Professor DUNHAM JACKSON: Fourier series and other special series. By Professor A. L. UNDERHILL: Differential equations. By Dr. ELIZABETH CARLSON: Advanced geometry. Second term: By Professor W. L. HART (topic to be announced later).

University of Oklahoma, June 10 to August 4. By Professor S. W. REAVES: Integral calculus; Modern geometry. By Professor J. O. HASSLER: Differential calculus; Theory of functions of a complex variable; Teachers course in mathematics. By Associate Professor E. D. MEACHAM: Analytic geometry; Projective geometry.

University of Pennsylvania, July 6 to August 15. In addition to the usual courses in Solid geometry, Trigonometry, College algebra, Analytic geometry, and Calculus, the following courses are offered: By Professor G. G. CHAMBERS: Elementary statistics. By Professor H. H. MITCHELL: Algebras and their arithmetics. By Professor J. R. KLINE: Integral equations.

University of Texas. By Professor E. L. DODD: Theory of irrational numbers; Probability and least squares. By Professor R. L. MOORE: Theory of sets; Introduction to the foundations of geometry. By Associate Professor H. J. ETTLINGER: Complex numbers and vector analysis; Infinite processes; Theory of the Riemann integral. By Associate Professor A. A. Bennett: Fundamentals in elementary mathematics; Ruler and compass constructions.

University of Wisconsin, June 29 to August 7. By Professor L. W. DOWLING: Projective geometry; Selected topics in higher geometry. By Professor H. W. MARCH: Definite integrals. By Professor W. W. HART: The teaching of secondary mathematics. By Professor M. H. INGRAHAM: Finite number fields; The elementary properties of number systems. By Professor W. WEAVER: Theory of probability; Vector analysis. By Dr. F. E. ALLEN: Analytic projective geometry. By Mr. E. B. MILLER: Differential equations; Theory of equations.

Mathematical Research will be directed by Professors DOWLING, MARCH, WEAVER, and INGRAHAM.

ELEVENTH REGULAR MEETING OF THE KANSAS SECTION.

The eleventh regular meeting of the Kansas Section was held at Topeka, February 7, 1925, in connection with the annual meeting of the Kansas Association of Mathematics Teachers. Two sessions were held of which the first was a joint session with the Kansas Association. Professor J. J. Wheeler, Chairman of the Kansas Section, presided part of the time during the joint (forenoon) session and all of the time during the afternoon session.

The attendance was forty-six including the following twenty-eight members of the Association:

C. H. Ashton, Wealthy Babcock, T. Bell, Florence Black, Lucy Dougherty, P. L. Evans, W. H. Garrett, W. A. Harshbarger, Emma Hyde, W. H. Hill, W. C. Janes, C. F. Lewis, Viola Lindberg, O. B. Loewen, W. A. Luby, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza Mossman, C. A. Reagan, B. L. Remick, J. A. G. Shirk, G. W. Smith, Edith Steininger, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, W. H. GARRETT; Vice-Chairman, W. T. STRATTON; Secretary-Treasurer, U. G. MITCHELL.

The following papers by members of the Association were presented:

(1) "Report of the Toronto meeting, International Mathematical Congress" by Professor J. A. G. SHIRK.

(2) "The evolution of our number system" by Professor C. F. LEWIS.

(3) "Books mathematics teachers should read" by Professor O. B. LOEWEN.

(4) "Invariants of determinants with linear elements" by Miss EDITH STEININGER.

(5) "Fermat and Mersenne numbers" by Professor U. G. MITCHELL.

(6) "Reduction of the formulæ for standard deviation and coefficient of correlation" by Professor A. E. WHITE.

(7) "Future policies of the Section," general discussion led by Professor J. J. WHEELER.

The first three of the above papers were read at the joint session.

Abstracts of the papers are given below, the numbers corresponding to the numbers in the list of titles above.

(1) Professor Shirk spoke of the varied program of the Mathematical Congress meeting held at Toronto, the number of papers and most important results presented. He emphasized, in particular, the evidence given of new fields of employment for mathematicians.

(2) Professor Lewis's paper traced the historical development of the real number system, with special attention to the development of various notations.

(3) Professor Loewen discussed certain types of books which should be of especial interest to teachers of mathematics, mentioning a number of valuable